Problem Set 2, 18.100B/C, Fall 2011

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1

Problem 15 from page 23.

Let $a = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_n) \in \mathbb{C}^n$. Put

$$A = \sum_{i=1}^{n} |a_j|^2,\ B = \sum_{j=1}^{n} |b_j|^2 \text{ and } C = \sum_{j=1}^{n} a_j \bar{b}_j.$$

Then it was proved in Theorem 1.35 that

$$B(AB - |C|^2) = \sum_{j=1}^{n} |Ba_j - Cb_j|^2 \geq 0.$$

The Cauchy-Schwarz inequality is

$$|C|^2 \leq AB.$$

When $B = 0$ we have $b = 0$; thus $C = 0$, $Ba_j = Cb_j$ for all j, and we have equality. When $B \neq 0$ we have equality if and only if

$$\sum_{j=1}^{n} |Ba_j - Cb_j|^2 = 0 \iff Ba_j = Cb_j \text{ for each } j \in \{1, \ldots n\}.$$

Thus, in all cases we have equality in the Cauchy Schwarz inequality if and only if $Ba_j = Cb_j$ for all j. Now, $Ba_j = Cb_j$ for all j is equivalent to $\lambda a = \mu b$ for some $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0\}$. You should check this.

2

Let X be a metric space and $E \subset X$.

a

Let $p \in E^\circ$. By definition, p is an interior point of E, so there exists an $r > 0$ such that $N_r(p) \subset E$. If we can show $N_r(p) \subset E^\circ$ it will follow that p is an interior point of E°, and thus E° is open. But for any $q \in N_r(p)$ we have

$$N_{r-d(p,q)}(q) \subset N_r(p) \subset E,$$

which implies $q \in E^\circ$, as required.

(For the above inclusion we use the triangle inequality:

$$x \in N_{r-d(p,q)}(q) \implies d(x,q) < r - d(p,q) \implies d(x,p) \le d(x,q) + d(p,q) < r \implies x \in N_r(p).)$$

b

E is open $\iff$ every point of E is an interior point of E $\iff$ $E \subset E^\circ$.

It is clear that we always have $E^\circ \subset E$ (since a neighborhood of a point contains the point). Hence, E is open if and only if $E^\circ = E$.

c

Let $G \subset E$ and suppose G is open. Given $p \in G$, there exists an $r > 0$ such that $N_r(p) \subset G$. Since $G \subset E$ we have

$$N_r(p) \subset E$$

and so $p \in E^\circ$.

d

By definition, $x \in E^\circ$ if and only if there exists an $r > 0$ such that $N_r(x) \subset E$. Thus, $x \notin E^\circ$ if and only if for all $r > 0$, $N_r(x) \cap (X \setminus E) \neq \emptyset$.

Suppose that for all $r > 0$, $N_r(x) \cap (X \setminus E) \neq \emptyset$. Then either $x \in X \setminus E$ or x is a limit point of $X \setminus E$, i.e. $x \in \overline{X \setminus E}$. Conversely, if $x \in \overline{X \setminus E}$, then either $x \in X \setminus E$ or x is a limit point of $X \setminus E$ and in either case $N_r(x) \cap (X \setminus E) \neq \emptyset$, for all $r > 0$.

e, f

No, in both cases. Let $X = \mathbb{R}$ and $E = \mathbb{Q}$.

Claim: $E^\circ = \emptyset$ and $\overline{E} = X$.

Proof: Let $x \in X$. Then for each $r > 0$, there exists a $q_r \in E$ with $x < q_r < x + r$. Thus

$$q_r \in (N_r(x) \setminus \{x\}) \cap E \neq \emptyset$$

for each $r > 0$. This says x is a limit point of E and so $x \in \overline{E}$, giving $\overline{E} = X$. Similarly, $\overline{X \setminus E} = X$ and so $X \setminus E^\circ = X$, which gives $E^\circ = \emptyset$.

One easily sees $\overline{E^\circ} = \emptyset$ and $(\overline{E})^\circ = X$ and so we have counterexamples.

3

a

Let $x, y \in \mathbb{R}^n$ and suppose $x \cdot y \geq 0$. We have

$$0 \leq (|x|y - |y|x) \cdot (|x|y - |y|x) = 2|x|^2|y|^2 - 2|x||y|x \cdot y$$

giving

$$|x \cdot y||x||y| = x \cdot y|x||y| \leq |x|^2|y|^2.$$

If $|x| = 0$ then $x = 0$ and $|x \cdot y| = |x||y|$. Similarly, if $|y| = 0$, we have $|x \cdot y| = |x||y|$. If $|x|, |y| \neq 0$ then we can divide through by $|x||y|$ to obtain

$$|x \cdot y| \leq |x||y|.$$

Also, in this case we see that we have equality if and only if $|x|y = |y|x$. In any case, we have equality if and only if x and y lie on a line through the origin.

If $x \cdot y \leq 0$ we can obtain the same result by proceding with $(|x|y + |y|x)$ in place of $(|x|y - |y|x)$.

b

Let $a_1, \ldots, a_n$ be positive real numbers. Setting $x = (\sqrt{a_1}, \ldots, \sqrt{a_n})$ and $y = (1/\sqrt{a_1}, \ldots, 1/\sqrt{a_n})$, the above inequality gives

$$n^2 = |x \cdot y|^2 \leq |x|^2|y|^2 = (a_1 + \ldots + a_n)\,(1/a_1 + \ldots + 1/a_n)$$

Thus the result is immediate and we have equality if and only if $M = (\sum a_i)\,(\sum 1/a_i)$ and x and y lie on a line through the origin. The latter condition holds if and only if there exists a $\lambda \in \mathbb{R}$ such that $\sqrt{a_i} = \lambda/\sqrt{a_i}$ for all i, i.e. $a_1 = \ldots = a_n$.

4

A set is infinite in the sense of 2.4 if and only if it is in bijection with a proper subset of itself:

a

Lemma: Let $\emptyset \neq X \subset J_N$. Then X has a maximum element.

Define a function

$$f : J_N \longrightarrow J_N, \;\; x \longmapsto (N+1) - x$$

One easily checks that f is well-defined. Also $f(f(x)) = x$ so that f is a bijection. Let i be the least element of $f(X) \neq \emptyset$. Then $f(i)$ is the maximum element of X: check this.

Lemma: Let $n \in \mathbb{N}$ and suppose $\emptyset \neq X \subsetneq J_n$. Then there exists a $k \in \mathbb{N}$ such that $k < n$ and a bijection

$$f : J_k \longrightarrow X.$$

Proof: We induct on n. When $n = 1$ there is nothing to prove since $J_1 = \{1\}$ has no nonempty proper subsets. If it makes you feel happier we can start the induction with the $n = 2$ case: the only nonempty proper subsets of $J_2 = \{1, 2\}$ are $\{1\}$ and $\{2\}$, which are both in bijection with J_1.

Since $J_n \setminus X \neq \emptyset$ we may let $m = \max(J_N \setminus X)$. Define a function

$$g : X \longrightarrow J_{N-1}, \quad i \longmapsto \begin{cases} i & \text{when } i < m \\ i - 1 & \text{when } i > m \end{cases}$$

One easily checks that g is injective and induces a bijection

$$X \longrightarrow g(X).$$

If $g(X) = J_{N-1}$, we are done: we may take $f = g$. Otherwise, $\emptyset \neq g(X) \subsetneq J_{n-1}$ and an inductive hypothesis tells us that $g(X)$ is in bijection with some J_k where $k < n - 1$. Let f be the composite of g and this bijection.

b

Lemma: Suppose $f : J_k \longrightarrow J_l$ is an injection; then $k \leq l$.

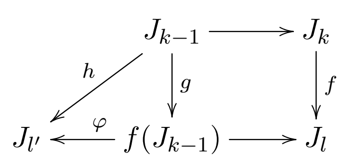
Proof: We induct on k. When $k = 1$ the result is clear.

Suppose inductively that whenever $f : J_{k-1} \longrightarrow J_l$ is an injection, $k - 1 \leq l$. Now suppose we are given an injection $f : J_k \longrightarrow J_l$. Define

$$g : J_{k-1} \longrightarrow f(J_{k-1}), \; i \longmapsto f(i).$$

Injectivity of f implies that g is injective, too.

Aside: If we draw the maps f and g vertically and the inclusions $J_{k-1} \subset J_k$, $f(J_{k-1}) \subset J_l$ horizontally we obtain the square, which constitutes the right hand section of the diagram below. By definition, the compositions of the functions as we go around the square in the two different possible ways are equal; we say the square commutes.



Suppose for contradiction that $f(J_{k-1}) = J_l$. Then there exists an $i \in J_{k-1}$ such that $f(i) = f(k)$. This contradicts our assumption that f is injective. Thus, $f(J_{k-1})$ is a proper subset of J_l. By (a), there exists a bijection $\varphi : f(J_{k-1}) \longrightarrow J_{l'}$, where $l' < l$. Thus $h = \varphi \circ g$, is an injection and our inductive hypothesis tells us that $k - 1 \leq l' < l$, which gives $k \leq l$, as required.

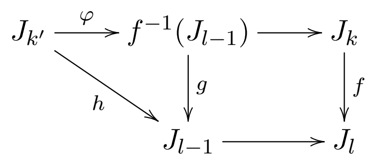
Lemma: Suppose $f : J_k \longrightarrow J_l$ is a surjection; then $k \geq l$.

Proof: We induct on l. When $l = 1$ the result is clear.

Suppose $f : J_k \longrightarrow J_l$ is a surjection, where $l > 1$. Then $f^{-1}(J_{l-1}) \neq \emptyset$ and we can define

$$g : f^{-1}(J_{l-1}) \longrightarrow J_{l-1}, \ i \longmapsto f(i).$$

Surjectivity of f implies that g is surjective, too. Also, because f is surjective, there exists an $i \in J_k$ such that $f(i) = l$. Then $i \notin f^{-1}(J_{l-1})$ and $f^{-1}(J_{l-1})$ is a proper subset of J_k. By (a), there exists a bijection $\varphi : J_{k'} \longrightarrow f^{-1}(J_{l-1})$, where $k' < k$.



$h = g \circ \varphi$ is a surjection and by an inductive hypothesis $k' \geq l - 1$. Thus $k \geq l$, as required.

Lemma: Suppose $f : J_n \longrightarrow J_n$ is injective; then f is surjective.

Proof: Suppose for contradiction that f is not surjective. Then $f(J_n)$ is a proper subset of J_n. Thus, by (a) we have a bijection $\varphi : f(J_n) \longrightarrow J_k$ where $k < n$.

$$g : J_n \longrightarrow J_k, \ i \longmapsto \varphi(f(i))$$

is injective. Thus $n \leq k$ by the first lemma, a contradiction.

c

Let X be finite nonempty subset and $\emptyset \neq Y \subsetneq X$. By definition, we have a bijection $f : X \longrightarrow J_n$ for some $n \in \mathbb{N}$. $\emptyset \neq f(Y) \subsetneq J_n$ and so $f(Y)$ is in bijection with J_k for some $k < n$, by (a). A bijection between X and Y would give a bijection between J_n and J_k, which is not possible, by (b).

d

Define

$$f : \mathbb{N} \longrightarrow \{n \in \mathbb{N} : n > 1\}, \ n \longmapsto n + 1.$$

Then f is a bijection between $\mathbb{N}$ and a proper subset of $\mathbb{N}$. By (c), $\mathbb{N}$ must be infinite.

e

Since X is infinite $X \neq \emptyset$ and we can choose $x \in X$. Define

$$f_1 : J_1 \longrightarrow X, \ 1 \longmapsto x.$$

Suppose inductively that we have an injection

$$f_{n-1} : J_{n-1} \longrightarrow X.$$

Since X is infinite, f_{n-1} is not a bijection; it must fail to be surjective and we can pick $x' \in X \setminus f_{n-1}(J_{n-1})$. Define

$$f_n : J_n \longrightarrow X, \quad i \longmapsto \begin{cases} f_{n-1}(i) & \text{when } i < n \\ x' & \text{when } i = n \end{cases}$$

One easily sees that f_n is injective.

Define

$$f : \mathbb{N} \longrightarrow X, \ \ n \longrightarrow f_n(n)$$

Suppose $n \leq m$ and $f(n) = f(m)$. Then $f_m(n) = f_n(n) = f(n) = f(m) = f_m(m)$, so injectivity of f_m tells us that $n = m$. Thus f is an injection and we're done.

Conclusion

(c) tells us no finite set can be in bijection with a proper subset of itself. Given an infinite set X, take an injection $f : \mathbb{N} \longrightarrow X$ and define

$$g : X \longrightarrow X \setminus \{f(1)\}, \quad x \longmapsto \begin{cases} x & \text{when } x \notin f(\mathbb{N}) \\ f(n+1) & \text{when } x = f(n) \end{cases}$$

Injectivity of f means g is well-defined, and one easily checks g is a bijection.

5

a

Let $A = \{s \in S : s < f(s)\}$.

If $A = \emptyset$ then for all $s \in S$ we have $s \geq f(s)$; we must have $\inf S \leq f(\inf S)$, so $f(\inf S) = f(\inf S)$ and we're done.

If $A \neq \emptyset$ we may set $a = \sup A$. Let $s \in A$. Then $s \leq a$ and so by the definition of A and monotonicity of f we have $s < f(s) \leq f(a)$. This means $f(a)$ is an upper bound for A and by the definition of supremum we have $a \leq f(a)$.

If $a = f(a)$ we're done so suppose that $a < f(a)$. Then $f(a)$ is strictly greater than an upper bound for A, which means $f(a) \notin A$. By monotonicity of f we obtain $f(a) \leq f(f(a))$. Thus $f(a) = f(f(a))$ and we're done.

b

$$X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$$

We have a bijection

$$f : \mathbb{N} \longrightarrow X, \quad n \longmapsto \begin{cases} \frac{1}{n-1} & \text{when } n > 1 \\ 0 & \text{when } n = 1 \end{cases}$$

Given a subset $S \subset \mathbb{N}$ such that $S \setminus \{1\} \neq \emptyset$ we have $\sup f(S) = f(\min(S \setminus \{1\})) \in X$; also, $\sup\{0\} = 0 \in X$. If $T \subset X$ is finite then certainly the inf is obtained. If $T \subset X$ is infinite then $\inf T = 0 \in X$.

c

$$[0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$$

Any nonempty subset of $[0,1]$ has 0 as a lower bound and 1 as an upper bound and so has an infimum and supremum in $[0,1]$.

6

First note that the set of all algebraic integers contains the integers, and so is an infinite set.

Let S be the set

$$\mathbb{N} \times \prod_{n=0}^{\infty} \mathbb{Z} = \{(n, a_0, a_1, a_2, \dots) : n \in \mathbb{N} \text{ and } a_j \in \mathbb{Z} \text{ for all } j \in \mathbb{N} \cup \{0\}\}$$

Let S_N be the subset

$$\{(n, a_0, a_1, \dots) \in S : n + |a_0| + |a_1| + \dots |a_n| = N,\ n < N,\ a_j = 0 \text{ for } j > n\}$$

The set of algebraic numbers is equal to

$$\bigcup_{N=1}^{\infty} \bigcup_{(n,a_0,a_1,\dots) \in S_N} \{z \in \mathbb{C} : a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0\}$$

We are told in the question that each $\{z \in \mathbb{C} : a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0\}$ has size $\leq n$ and each S_N has size $\leq (2N+1)^{N+1}$. Thus

$$\bigcup_{(n,a_0,a_1,\dots) \in S_N} \{z \in \mathbb{C} : a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0\}$$

is finite for each N.

The answer now follows, since a countable union of finite sets is at most countable, for if $A_1, A_2, A_3, \dots$ is a sequence of finite sets we can define a surjection by

$$f : \mathbb{N} \longrightarrow \bigcup_{k=1}^{\infty} A_k, \quad i \longmapsto f_j(i - \sum_{k=1}^{j-1} |A_k|), \text{ when } \sum_{k=1}^{j-1} |A_k| < i \leq \sum_{k=1}^{j} |A_k|$$

after choosing bijections $f_j : J_{|A_j|} \longrightarrow A_j$ for each $j \in \mathbb{N}$.