Problem Set 10, 18.100B/C, Fall 2011

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1

Let $K : [0,1]^2 \longrightarrow \mathbb{R}$ and $g : [0,1] \longrightarrow [-1,1]$ be continuous. Then

$$\begin{aligned}\left|\int_0^1 K(x,y)g(y)dy - \int_0^1 K(\tilde{x},y)g(y)dy\right| &\leq \int_0^1 |K(x,y) - K(\tilde{x},y)|\,|g(y)|\,dy \\ &\leq \sup_{y\in[0,1]} |K(x,y) - K(\tilde{x},y)|\end{aligned}$$

Suppose we are given $\epsilon > 0$. Since K is continuous on a compact set K is uniformly continuous. Thus there exists $\delta > 0$ such that

$$|x - \tilde{x}| < \delta \implies \sup_{y\in[0,1]} |K(x,y) - K(\tilde{x},y)| < \epsilon,$$

which shows that the family $\mathcal{F}$ considered in the question is equicontinuous.

2

Suppose we are given $\epsilon > 0$. Since (f_n) is a sequence of equicontinuous functions there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$,

$$d(x,y) < \delta \implies d(f_n(x), f_n(y)) < \epsilon/3.$$

Since K is compact there exists $x_1, \ldots, x_r \in K$ such that

$$\bigcup_{i=1}^{r} N_\delta(x_i) = K.$$

For each $i \in \{1, \ldots, r\}$, $(f_n(x_i))$ converges and is therefore Cauchy. So there exists an $N \in \mathbb{N}$ such that for each $i \in \{1, \ldots, r\}$

$$n, m \geq N \implies d(f_n(x_i), f_m(x_i)) < \epsilon/3.$$

Let $x \in K$. Choose $i \in \{1, \ldots, r\}$ such that $d(x, x_i) < \delta$. Then

$$n, m \geq N \implies d(f_n(x), f_m(x)) \leq d(f_n(x), f_n(x_i)) + d(f_n(x_i), f_m(x_i)) + d(f_m(x_i), f_m(x)) < \epsilon.$$

Thus

$$n, m \geq N \implies \sup_{x \in K} d(f_n(x), f_m(x)) \leq \epsilon.$$

This shows that (f_n) converges uniformly by the Cauchy criterion.

3

Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous and suppose that $\int_0^1 x^n f(x)dx = 0$ for all $n \in \mathbb{N}$. From linearity of the integral we deduce that $\int_0^1 p(x)f(x)dx = 0$ for all polynomials p. Weierstrass' theorem says that we can find a sequence of polynomials (p_n) converging uniformly to f. Since f is bounded $(p_n f)$ converges uniformly to f^2. Then

$$\int_0^1 p_n(x)f(x)dx \longrightarrow \int_0^1 f(x)^2 dx$$

giving $\int_0^1 f(x)^2 dx = 0$. Since f^2 is positive and continuous this implies that $f^2 = 0$ and so $f = 0$.

4

Let X be a metric space with metric d. Fix a point $a \in X$. For $p \in X$ let

$$f_p(x) = d(x,p) - d(x,a).$$

$d(x,p) \leq d(x,a) + d(a,p)$ implies $f_p(x) \leq d(a,p)$ and $d(x,a) \leq d(x,p) + d(a,p)$ implies $-f_p(x) \leq d(a,p)$ so that $|f_p(x)| \leq d(a,p)$. Similarly, $|f_p(x) - f_p(y)| = |d(x,p) - d(y,p)| \leq d(x,y)$ which implies f_p is continuous. Hence, $f_p \in \mathscr{C}(X)$.

$|f_p(x) - f_q(x)| = |d(x,p) - d(x,q)| \leq d(p,q)$ and $f_p(q) - f_q(q) = d(p,q)$ so that $\|f_p - f_q\| = d(p,q)$.

Y is complete as a closed subset of a complete space.

5

Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

We may prove by induction that there exists polynomials $p_n(x)$ and integers $r_n \in \mathbb{N}$ such that for $x \neq 0$

$$f^{(n)}(x) = x^{-r_n} p_n(x) f(x).$$

Suppose inductively that $f^{(n)}(0) = 0$, where $n \in \mathbb{N} \cup \{0\}$. Then

$$f^{(n+1)}(0) = \lim_{x \longrightarrow 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x \longrightarrow 0} \frac{f^{(n)}(x)}{x} = \lim_{x \longrightarrow 0} x^{-r_n - 1} p_n(x) f(x) = 0$$

since exponential decay kills polynomial growth.

6

(The question is wrongly stated.)

Let $f : [0,1]^2 \longrightarrow \mathbb{R}$ be continuous and suppose that $\frac{\partial f}{\partial x} : [0,1]^2 \longrightarrow \mathbb{R}$ exists and is continuous. By definition

$$\frac{d}{dx}\int_0^1 f(x,y)dy = \lim_{h\longrightarrow 0}\int_0^1 \frac{f(x+h,y)-f(x,y)}{h}dy$$

and

$$\int_0^1 \frac{\partial f}{\partial x}(x,y)dy = \int_0^1 \lim_{h\longrightarrow 0}\frac{f(x+h,y)-f(x,y)}{h}dy.$$

Thus in order to prove

$$\frac{d}{dx}\int_0^1 f(x,y)dy = \int_0^1 \frac{\partial f}{\partial x}(x,y)dy$$

it is enough to argue that $\frac{f(x+h,y)-f(x,y)}{h}$ converges to $\frac{\partial f}{\partial x}(x,y)$, uniformly in y, as $h \longrightarrow 0$. This is true because the mean value theorem gives a $\xi_{x,h,y} \in (x-h, x+h)$ such that $\frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x}(\xi_{x,h,y}, y)$ and because $\frac{\partial f}{\partial x}(x,y)$ is uniformly continuous.