Problem Set 6

Due October 27th at 4 pm in room 2-108.

Hand in parts 1, 2 and 3 separately. Put your name and whether you are registered for 18.100B or 18.100C on each part.

Part 1

1. Problem 9 from page 79.
2. Suppose $(a_n)_{n\in\mathbb{N}}$ satisfies $a_{n+m} \leq a_n + a_m$ for all $m, n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf\left\{\frac{a_n}{n} : n \in \mathbb{N}\right\},$$

as an element of $\mathbb{R} \cup \{-\infty\}$. You may use without proof the *Euclidean division algorithm*, which says that for any $n, \ell \in \mathbb{N}$, there exist unique $m, r \in \mathbb{N} \cup \{0\}$ with $r < \ell$ such that

$$n = m\ell + r.$$

Part 2

3. If $(a_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers satisfying

$$a_{n+1} \leq a_n + \frac{1}{n^2}, \qquad \forall n \in \mathbb{N}$$

prove that $(a_n)_{n\in\mathbb{N}}$ converges.

4. Prove that if $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that $\sum |a_{n+1} - a_n|$ converges, then $(a_n)_{n\in\mathbb{N}}$ converges. Prove that the converse does not hold.
5. Prove that $(a_n)_{n\in\mathbb{N}}$ converges if and only if $(2a_{n+1} - a_n)_{n\in\mathbb{N}}$ does.

Part 3

6. Problem 4 from page 98. Students registered for 18.100C should write this problem up in LaTeX.
7. Problem 23 from page 101.