Problem Set 5

Due October 20th at 4 pm in room 2-108.

Hand in parts 1, 2 and 3 separately. Put your name and whether you are registered for 18.100B or 18.100C on each part.

Part 1

1. (a) Prove that if $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence of real numbers, then

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup\{a_m \colon m \geq n\}\right).$$

   (b) Prove that if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded sequences of real numbers, then

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

   and that equality holds if $(a_n)_{n\in\mathbb{N}}$ converges. Give an example of two sequences for which equality does not hold.

2. (a) If $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and if $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ converge and have the same limit, prove that $(b_n)_{n\in\mathbb{N}}$ also converges and also has the same limit.

   (b) Let $k \in \mathbb{N}$ and let $x_1 \geq x_2 \geq \cdots \geq x_k \geq 0$. Evaluate

$$\lim_{n\to\infty} \left(x_1^n + \cdots + x_k^n\right)^{1/n}.$$

Part 2

3. Prove that a sequence in a metric space converges to a point s if and only if every subsequence has a subsequence which converges to s. Students registered for 18.100C should write this problem up in LaTeX.
4. Problem 24 from page 82.

Part 3

5. Problem 6 from page 78.

6. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers which tends to zero but such that $\sum_{n=1}^{\infty} a_n$ diverges. Let $(A_n)_{n\in\mathbb{N}}$ be the sequence of partial sums

$$A_n = \sum_{k=1}^{n} a_k,$$

and let $b_{n+1} = \sqrt{A_{n+1}} - \sqrt{A_n}$. Show that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0,$$

but that $\sum_{n=1}^{\infty} b_n$ is still divergent. In this sense there is no 'smallest' divergent series, and one can similarly show that there is no 'largest' convergent one.