18.100B and 18.100C
Fall 2011

Problem Set 3

Due September 29th at 4 pm in room 2-108.

Hand in parts 1, 2 and 3 separately. Put your name and whether you are registered for 18.100B or 18.100C on each part.

Part 1

1. Let X be a metric space and $E \subset X$. Let $\mathrm{cl}(E)$ denote the closure of E and let $\mathrm{int}(E)$ denote the interior of E.
   (a) Prove that $\mathrm{int}(\mathrm{int}(E)) = \mathrm{int}(E)$ and that $\mathrm{cl}(\mathrm{cl}(E)) = \mathrm{cl}(E)$.
   (b) Prove that $\mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(E)))) = \mathrm{int}(\mathrm{cl}(E))$, and deduce from this that $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(E)))) = \mathrm{cl}(\mathrm{int}(E))$.
   (c) Conclude that beginning with any subset E of a metric space X, at most seven distinct sets can be obtained by taking successive interior and closure operations.
   (d) Give an example of a subset of $\mathbb{R}$ (equipped with the Euclidean metric) for which these seven sets are indeed distinct.
2. Let $n \in \mathbb{N}$ and let $S \subset \mathbb{R}^n$ be a set such that every point in S is isolated. Show that S is at most countable. Students registered for 18.100C should write this problem up in LaTeX.

Part 2

3. For each of the following subsets of $\mathbb{R}$, determine whether the set is open, whether it is closed, and whether it is compact. Also, find the interior, the limit points and the closure. For this problem you do not need to provide any proofs.
   (a) $\{1, 2, 3\}$.
   (b) $[-1, 0) \cup (0, 1]$.
   (c) $\mathbb{Q}$.
   (d) The complement of $\mathbb{Q}$.

   Do the same thing for the following subsets of $\mathbb{R}^2$.
   (e) $\{(x, y) \in \mathbb{R}^2 \colon y > 0\}$.

(f) $\{(x, y) \in \mathbb{R}^2 \colon x \in [-1, 0) \cup (0, 1]\}$.

4. Problem 13 from page 44.

Part 3

5. Let K be a compact metric space, and $\varepsilon > 0$. Show that there exists $N \in \mathbb{N}$ such that every set of N distinct points in K includes at least two points with distance less than ε between them.
6. Let K be a compact metric space. Show that K has a subset which is dense and at most countable.
7. Let K be a compact metric space and $f \colon K \to K$ a function satisfying $d(f(x), f(y)) = d(x, y)$ for all $x, y \in K$. Prove that $f(K) = K$.

   Hint: First prove that $f(K)$ is closed. Then suppose $f(K) \neq K$, and take a point $x \in f(K)^c$. Look at the limit points of the set

   $$\{f^n(x) \colon n \in \mathbb{N}\},$$

   where $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, and so on (these are called the *iterates* of x under f). What can you say about $d(f^m(x), f^n(x))$ for $m, n \in \mathbb{N}$?