18.100B and 18.100C
Fall 2011

Problem Set 2

Due September 22nd at 4 pm in room 2-108.

Hand in parts 1, 2 and 3 separately. Put your name and whether you are registered for 18.100B or 18.100C on each part.

Part 1

1. Problem 15 from page 23. Students registered for 18.100C should write this problem up in LaTeX.

2. Problem 9 from page 43.

Part 2

3. (a) Give an alternate proof that $|x \cdot y| \leq |x||y|$ for any $x, y \in \mathbb{R}^n$ and for any $n \in \mathbb{N}$ by taking the inner product of the vector $|x|y - |y|x$ with itself and using the fact that this is nonnegative.

   (b) Let $a_1, \ldots, a_n$ be positive real numbers. Prove that if
$$(a_1 + \cdots + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) \leq M$$
   for some $M > 0$, then $n \leq \sqrt{M}$. When does equality hold?

4. Prove that a set is infinite in the sense of §2.4 if and only if it is in bijection with a proper subset of itself by proving the following statements:

   (a) If X is a proper subset of J_n for some $n \in \mathbb{N}$, then either X is empty or X is in bijection with J_k for some $k \in \mathbb{N}$, $k < n$.

   (b) If $f\colon J_k \to J_\ell$ is an injection, then $k \leq \ell$. Moreover, $k = \ell$ if and only if f is also a surjection.

   (c) No finite set can be in bijection with a proper subset of itself.

   (d) $\mathbb{N}$ is infinite.

   (e) If X is infinite, there is an injection $f\colon \mathbb{N} \to X$.

   Finally, use (c), (d), and (e) to conclude.

Part 3

5. Let S be a nonempty ordered set with the property that every nonempty subset of S has both a least upper bound in S and a greatest lower bound in S.

   (a) Prove that if $f\colon S \to S$ satisfies $f(x) \leq f(y)$ for any $x, y \in S$ with $x \leq y$, then there exists an $x_0 \in S$ for which $f(x_0) = x_0$.

   (b) Give an example of an infinite subset of $\mathbb{Q}$ with this property.

   (c) Give an example of an uncountable subset of $\mathbb{R}$ with this property.

6. Problem 2 from page 43. You may use without proof the fact that if $a_0, \ldots a_n$ are integers which are not all zero, then there are at most n complex solutions to $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$.