18.100B and 18.100C
Fall 2011

Second midterm review sheet

The second midterm covers all the material from chapters 3 – 5, and you may use results from chapters 1 – 5 without proof. Most of the midterm will consist of problems from this list. You will not be asked to reprove theorems from the book.

1. Evaluate
$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}.$$
2. Let $p > 1$. Evaluate
$$\frac{\displaystyle\sum_{n=1}^{\infty} n^{-p}}{\displaystyle\sum_{n=1}^{\infty} (-1)^n n^{-p}}.$$
3. Let $A > 0$, and suppose $\sum_{n=1}^{\infty} a_n = A$, and $a_n > 0$ for all $n \in \mathbb{N}$. Find the possible values of $\sum_{n=1}^{\infty} a_n^2$.
4. Let $\sum b_n$ be a convergent series of real numbers, and let $(a_n)_{n\in\mathbb{N}}$ be bounded below. Prove that if
$$a_{n+1} \le a_n + b_n,$$
then $(a_n)_{n\in\mathbb{N}}$ converges.
5. Problem 18 from page 100.
6. Problem 25 from page 102.
7. Let X and Y be metric spaces and f a function from X to Y. Prove that the following are equivalent.
   (a) f is continuous.
   (b) If $(p_n)_{n\in\mathbb{N}}$ is a convergent sequence in X, then the sequence $(f(p_n))_{n\in\mathbb{N}}$ is convergent in Y.
   (c) If $(p_n)_{n\in\mathbb{N}}$ is a convergent sequence in X, then the sequence $(f(p_n))_{n\in\mathbb{N}}$

is convergent in Y and

$$\lim_{n\to\infty} f(p_n) = f\left(\lim_{n\to\infty} p_n\right).$$

8. Let K be a compact metric space and $A \subset K$. Prove that A is compact if and only if for every continuous function $f\colon K \to \mathbb{R}$, the restriction of f to A attains a maximum on A.
9. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty.$$

   Prove that f attains a minimum on $\mathbb{R}$.
10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous with $f(0) = 0$. Prove that there exists $M > 0$ such that $|f(x)| \leq 1 + M|x|$ for all $x \in \mathbb{R}$.
11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period 1 (meaning that it obeys $f(x+1) = f(x)$ for all $x \in \mathbb{R}$)
    (a) Prove that f is bounded and attains both a maximum and a minimum on $\mathbb{R}$.
    (b) Prove that f is uniformly continuous on $\mathbb{R}$.
    (c) Prove that for any $a \in \mathbb{R}$ there is $x_0 \in \mathbb{R}$ such that $f(x_0+a) = f(x_0)$.
12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Prove that the set

$$\{\alpha \in \mathbb{R}\colon \exists (x_n)_{n\in\mathbb{N}} \text{ with } \lim_{n\to\infty} x_n = \infty,\ \lim_{n\to\infty} f(x_n) = \alpha\}$$

    is connected.
13. Problem 2 from page 114.
14. Problem 6 from page 114
15. Problem 11 from page 115.
16. Problem 14 from page 115.