18.100B and 18.100C
Fall 2011

First midterm review sheet

The first midterm covers all the material from chapters one and two, except for the appendix to chapter one. On the midterm you will be asked to give the proof, or a part of the proof, of one or two of these theorems: 1.11, 1.20, 2.12, 2.14, 2.19, 2.23, 2.24, 2.28, 2.34, 2.35, 2.36, 2.37.

You will also be asked to solve several of the following problems. You may use results from chapters 1 and 2 of Rudin without proof. You may not use results from problem sets or from the exercises at the ends of the chapters. Note that these problems are not arranged in order of difficulty!

1. Problem 9 from page 22.
2. Give an alternative proof that $|x \cdot y| \leq |x||y|$ for any $x, y \in \mathbb{R}^n$ and for any $n \in \mathbb{N}$ by using the fact that the dot product of $x + ty$ with itself is nonnegative for any $t \in \mathbb{R}$. (*Hint:* Complete the square.)
3. Let A be a nonempty set, and let $\mathcal{P}(A)$ be the set of subsets of A. Prove that the cardinality of $\mathcal{P}(A)$ is strictly greater than that of A using the following two steps:
   (a) Give an example of an injective function $A \to \mathcal{P}(A)$.
   (b) Let $f\colon A \to \mathcal{P}(A)$ be any function. Use the set $\{x \in A\colon x \notin f(x)\}$ to show that f is not surjective.

   Conclude that A is in bijection with a proper subset of $\mathcal{P}(A)$ but not with $\mathcal{P}(A)$ itself.

   If A is a set of n elements for some $n \in \mathbb{N}$, compute the number of elements in $\mathcal{P}(A)$ by constructing a bijection from $\mathcal{P}(A)$ to $\{0,1\}^A$, where $\{0,1\}^A$ is defined to be the set of functions from A into $\{0,1\}$. Finally, use this bijection, together with Cantor's diagonal process (see page 30), to give another proof that there is no surjection $f\colon A \to \mathcal{P}(A)$, even if A is infinite.
4. Show that the open interval $(0,1) = \{x \in \mathbb{R}\colon 0 < x < 1\}$ is in bijection with the open square $(0,1) \times (0,1) = \{(x,y) \in \mathbb{R}^2\colon 0 < x < 1, 0 < y < 1\}$. Then show that $\mathbb{R}$ is in bijection with $\mathbb{R}^2$ by showing that it is in bijection with $(0,1)$. Cantor was the first to discover this, and he was very surprised and wrote letters to some friends sharing his excitement.
5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Prove that there is a countable subfield $K \subset \mathbb{R}$ such that $f(K) \subset K$.

6. Problem 7 from page 43.
7. If E is a subset of a metric space X, let $L(E)$ denote the set of limit points of E. Let $L^2(E) = L(L(E))$, let $L^3(E) = L(L(L(E)))$, and so on. Let $L^0(E) = E$. The set $L^n(E)$ is called the n^{th} *derived set* of E.

   Show that for every $n \in \mathbb{N}$ there exists $E \subset \mathbb{R}$ such that $L^{n-1}(E) \neq \varnothing$ but $L^n(E) = \varnothing$.
8. Prove that a finite union of compact sets is compact, but that a countable union need not be.
9. Problem 15 from page 44.
10. Let $G \subset \mathbb{R}^2$ be open, and suppose that

$$\{(x, y) \in \mathbb{R}^2 \colon x \in [0, 1], y \in [0, 1]\} \subset G.$$

   Prove that there exists $\varepsilon > 0$ such that

$$\{(x, y) \in \mathbb{R}^2 \colon x \in [0, 1 + \varepsilon], y \in [0, 1]\} \subset G.$$

11. Problem 20 from page 44.
12. Let X be a metric space. Prove that the following are equivalent:
    (a) X is connected.
    (b) The only subsets of X which are both open and closed are X and $\varnothing$.
    (c) For every two points $x, y \in X$, there is a connected subset $E \subset X$ containing both x and y.
13. Let X be a compact metric space, and let $K_1, K_2, \ldots$ be a sequence of subsets of X which are nonempty, closed, and connected, and which satisfy $K_{n+1} \subset K_n$ for every $n \in \mathbb{N}$. Prove that $\cap_{n=1}^{\infty} K_n$ is connected.
14. Problem 30 from page 46.

Finally, you will also be asked to solve one or two problems not taken from this list.