18.100B and 18.100C
Fall 2011

Final exam review sheet

Most of the final will consist of problems from this list.

1. Prove that $\mathbb{R}^2$ is not the union of a countable family of lines.

2. Problem 10 from page 44.

3. Let $E \subset \mathbb{R}$ be uncountable.
   (a) Prove that E has uncountably many limit points.
   (b) Prove that there exists $x \in \mathbb{R}$ such that $E \cap (-\infty, x)$ and $E \cap (x, \infty)$ are both uncountable.

4. Let X be a metric space and $(K_n)_{n\in\mathbb{N}}$ a sequence of nonempty compact subsets of X with $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$. Prove that if U is an open set with $\bigcap_{n=1}^{\infty} K_n \subset U$, then there exists $N \in \mathbb{N}$ such that $K_N \subset U$.

5. Problem 8 from page 99.

6. Let $f\colon [0,1] \times [0,1] \to \mathbb{R}$ be continuous, and let $g\colon [0,1] \to \mathbb{R}$ be defined by
$$g(x) = \max\{f(x,y)\colon y \in [0,1]\}.$$
Prove that g is continuous.

7. Let K be a nonempty compact metric space with metric d, and suppose $f\colon K \to K$ obeys $d(f(x), f(y)) < d(x,y)$ for all $x, y \in K$ with $x \neq y$. Prove that there is a unique $x_0 \in K$ with $f(x_0) = x_0$.

8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $f^{-1}(E) \subset \mathbb{R}^2$ is bounded whenever $E \subset \mathbb{R}$ is bounded. Prove that f attains either a maximum or a minimum value.

9. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nonconstant, nondecreasing function. Prove that for any $R > 0$ there exist $a \in \mathbb{R}$ and $c > 0$ such that
$$f(a+x) - f(a-x) \geq cx,$$
for all $x \in [-R, R]$.

10. Let X be a metric space. A subset $E \subset X$ is *path-connected* if for every $p, q \in E$, there is a continuous function $c\colon [0,1] \to E$ with $c(0) = p$ and $c(1) = q$.
   (a) Prove that path-connected sets are connected.
   (b) Prove that open connected sets are path-connected.
   (c) Give an example of a subset of $\mathbb{R}^2$ which is connected but not path-connected.

11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is surjective.

12. Problem 3 from page 114.

13. Let $f\colon (-1,1) \to \mathbb{R}$ be differentiable, and suppose $f(x)/x^2$ has a finite limit as $x \to 0$. Does it follow that $f''(0)$ exists?

14. Let $f\colon (0,\infty) \to (0,\infty)$ be twice differentiable with f'' is bounded and continuous, and such that $f'(x) \leq 0$ for all $x > 0$. Prove that $\lim\limits_{x\to\infty} f'(x) = 0$

15. Let $f\colon [0,1] \to \mathbb{R}$ be differentiable with continuous derivative, and with $f(0) = 0$. Prove that

$$\sup_{0\leq x\leq 1} |f(x)| \leq \sqrt{\int_0^1 (f'(x))^2 dx}.$$

16. Let K be a compact metric space, and let $\{f_n\}_{n\in\mathbb{N}}$ be a uniformly bounded equicontinuous family of functions $K \to \mathbb{R}$. For each $n \in \mathbb{N}$, define $g_n\colon K \to \mathbb{R}$ by

$$g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}.$$

    Prove that the sequence $(g_n)_{n\in\mathbb{N}}$ converges uniformly.

17. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions $[0,1] \to \mathbb{R}$ such that the derivatives $\{(g_n')\}_{n\in\mathbb{N}}$ are uniformly bounded.
   (a) Show that there is a sequence of real numbers $(c_n)_{n\in\mathbb{N}}$ such that the sequence of functions $(h_n)_{n\in\mathbb{N}}$ defined by $h_n(x) = g_n(x) - c_n$ has a uniformly convergent subsequence.

(b) Show that if $(\int_0^1 g_n dx)_{n\in\mathbb{N}}$ is a bounded sequence of real numbers, then g_n has a uniformly convergent subsequence.

18. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $\mathbb{R}\to\mathbb{R}$ which converges uniformly to a function $f\colon \mathbb{R}\to\mathbb{R}$, and such that $c_n = \lim_{x\to\infty} f_n(x)$ exists for each $n\in\mathbb{N}$. Prove that $\lim_{n\to\infty} c_n$ and $\lim_{x\to\infty} f(x)$ both exist and are equal.

19. Let $N\in\mathbb{N}$. Prove that if f is a continuous function $[0,1]\to\mathbb{R}$ satisfying

$$\int_0^1 x^n f(x)dx = 0$$

for all $n\in\mathbb{N}$ with $n\geq N$, then $f\equiv 0$.

20. Problem 12 from page 140 and problem 22 from page 169.

21. Problem 27 from page 119 and problem 25 from page 170.

22. For $n\in\mathbb{N}$, let $I_n = \displaystyle\int_0^\pi \sin^n x dx$.
    (a) Evaluate I_n for each $n\in\mathbb{N}$.
    (b) Prove that $(I_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence of positive terms.
    (c) Evaluate the infinite product $\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdots$.

23. Problem 5 from page 197.

24. Problem 6 from page 197.

25. Problem 13 from page 198.