HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 10: DUE THURSDAY, May 3, AT 11:00 IN 2-108.

Please remember to tell us which lecture section you are in (Ciubotaru, Melrose, or Parker).

(1) (15 points) Assume f is a real, differentiable function with continuous derivative on $[a, b]$, $f(a) = f(b) = 0$, and $\int_a^b f^2(x)\, dx = 1$. Prove that
  (a) $\int_a^b x f(x) f'(x)\, dx = -\frac{1}{2}$, and
  (b) $\int_a^b [f'(x)]^2\, dx \cdot \int_a^b x^2 f^2(x)\, dx > \frac{1}{4}$.

(2) (15 points) Consider the series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}, \quad x > 0.$$

  (a) On what intervals does the series converges uniformly? On what intervals does it fail to converge uniformly?
  (b) Is f continuous wherever the series converges?

(3) (10 points) Define $f_n(x) = \frac{x}{1+nx^2}$, for $n \geq 1$. Prove that:
  (a) $\{f_n\}$ converges uniformly on $\mathbb{R}$ to some function f, and
  (b) $\{f_n'(x)\}$ converges pointwise to $\{f'(x)\}$ for all $x \neq 0$, but not for $x = 0$.

(4) (15 points) Recall the step function

$$I(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}.$$

Let $\{x_n\}$ be a sequence of distinct points in the interval (a, b), and $\sum c_n$ and absolutely convergent series. Prove that the series of functions

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n), \quad x \in [a, b]$$

converges uniformly.

In addition, show that f is continuous at every $x \neq x_n$.

(5) (15 points) Put $P_0 = 0$, and define, for $n \geq 0$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that the sequence of polynomials $\{P_n(x)\}_n$ converges uniformly to $|x|$ on the interval $[-1, 1]$. (Hint: see ex. 23/p. 169.)