18.100B/C. Test 2. April 26, 2007, 7:30-8:30PM

Problem 1. Suppose that f is Riemann integrable on $[0,1]$ with $\int_0^1 f dx = 1$. Show that there exists some $c \in (0,1)$ such that $\int_0^c f dx = \frac{1}{2}$.

Solution. Define $g(x) = \int_0^x f(t)dt$. By the fundamental theorem of calculus, g is continuous on $[0,1]$. Then g maps the connected set $[0,1]$ into a connected set, an interval, and since $g(0) = 0$, and $g(1) = 1$, there must exist $c \in (0,1)$ such that $g(c) = \frac{1}{2}$ (intermediate value property).

Problem 2. Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and bounded. Show that the set

$$\{(x,y) \in \mathbb{R}^2; f(x,y) = x^2 + y^2\}$$

is a compact subset of $\mathbb{R}^2$.

Solution. Define $g(x,y) = f(x,y) - (x^2+y^2)$. Then our set, call it E, is $E = g^{-1}(\{0\})$. Since f is continuous and polynomials are continuous, g is continuous (being a sum of continuous functions). Then $g^{-1}(\{0\})$ is closed (inverse image of a closed set).

Let M be such that $|f(x,y)| \leq M$ for all (x,y). If $(x,y) \in E$, then $x^2 + y^2 \leq M$, which means that E is a subset of the disk $x^2 + y^2 \leq M$, therefore bounded. So E is closed and bounded in $\mathbb{R}^2$, and by Heine-Borel it is compact.

Problem 3. Let X be a metric space with non-empty subsets $K \subset X$ compact and $F \subset X$ closed which are disjoint, $K \cap F = \emptyset$.

(a) Show that there exists $\epsilon > 0$ such that

$$d(p,q) > \epsilon \ \forall \ p \in K, \ q \in F.$$

(b) Give an example of two nonempty disjoint closed sets in a metric space for which this conclusion fails.

Solution. (a) *Proof 1*: Assume by contradiction that for every $\epsilon = \frac{1}{n}$, there exist $p_n \in K, q_n \in F$ such that $d(p_n, q_n) \leq \frac{1}{n}$. The sequence $\{p_n\} \subset K$ must have a convergent subsequence since K is compact. Let $\{p_{n_k}\} \to p \in K$ be this subsequence. From $d(p_{n_k}, q_{n_k}) \to 0$ and $d(p_{n_k}, p) \to 0$, as $k \to \infty$, we deduce that $\{q_{n_k}\} \subset F$ converges to p. Because, F is closed, this implies that $p \in F$, contradiction with $K \cap F = \emptyset$.

Proof 2: Define the function $d_F : K \to \mathbb{R}$, $d_F(p) = \inf_{q \in F} d(p,q)$. First, $d_F(p) > 0$, for all p. (Assume $d_F(p) = 0$, then there exists a sequence $\{q_n\} \subset F$, such that $\lim_{n\to\infty} d(p, q_n) = 0$. Since F is closed, it follows $p \in F$, contradiction with $K \cap F = \emptyset$.)

Secondly, d_F is continuous, in fact Lipschitz. Note that $d_F(p') = \inf_{q\in F} d(p', q) \leq \inf_{q\in F}(d(p',p) + d(p,q)) = d(p',p) + d_F(p)$. It follows that for every $p, p' \in K$, we have $|d_F(p) - d_F(p')| \leq d(p,p')$.

Since K is compact, $d_F(K)$ is compact. But $d_F(K) \subset (0, \infty)$, so there must exist $\epsilon > 0$, such that $d_F(K) \subset [\epsilon, \infty)$.

(b) Take F_1 to be the graph of e^x and F_2 to be the x-axis.

Problem 4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable at every point and satisfy

$$f(-10) = 1,\ f(0) = 0,\ f(10) = 1.$$

Show that there exists a point $x \in \mathbb{R}$ such that $f'(x) = \sqrt{2}/100$.

Solution. Apply the mean value theorem twice, on $[-10, 0]$, respectively $[0, 10]$. There exists points $x_1 \in (-10, 0)$ and $x_2 \in (0, 10)$, such that $f'(x_1) = -\frac{1}{10}$, and $f'(x_2) = \frac{1}{10}$. Now $-\frac{1}{10} < \frac{\sqrt{2}}{100} < \frac{1}{10}$, so by the intermediate value property of the derivative, there must exist $x \in (x_1, x_2)$, such that $f'(x) = \frac{\sqrt{2}}{100}$.

Problem 5. Let f and g be bounded real-valued functions on an interval $[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$. Let $\alpha : [a, b] \longrightarrow \mathbb{R}$ be a monotonic increasing function such that for every $\epsilon > 0$ there exist partitions $\mathcal{P}_-$ and $\mathcal{P}_+$ of $[a, b]$ such that

$$U(\mathcal{P}_+, g, \alpha) < L(\mathcal{P}_-, f, \alpha) + \epsilon.$$

Show that f and g are both Riemann-Stieltjes integrable with respect to α on $[a, b]$.

Solution. Let $\epsilon > 0$ be given. Since $f(x) \leq g(x)$, for every partition $\mathcal{P}$, we have $U(\mathcal{P}, f, \alpha) \leq U(\mathcal{P}, g, \alpha)$ and $L(\mathcal{P}, f, \alpha) \leq U(\mathcal{P}, g, \alpha)$.

Then $U(\mathcal{P}_+, g, \alpha) < L(\mathcal{P}_-, g, \alpha) + \epsilon$. Put $\mathcal{P}^* = \mathcal{P}_+ \cup \mathcal{P}_-$. Then

$$L(\mathcal{P}_-, g, \alpha) < L(\mathcal{P}^*, g, \alpha) < U(\mathcal{P}^*, g, \alpha) < U(\mathcal{P}_+, g, \alpha),$$

by the properties of the refinement. So $U(\mathcal{P}^*, g, \alpha) - L(\mathcal{P}^*, g, \alpha) < \epsilon$, which is the criterion for integrability for g.

The proof for f is the same.