18.100B/C TEST 1, MARCH 15, 2007
SOLUTIONS

You are not permitted to bring any books or other material with you. You may use theorems, lemmas and propositions from the book provided you can quote them correctly.

(1) (a) Show that any closed subset of a complete metric space is complete.
(b) Conversely, show that if (X, d) is a metric space, $E \subset X$ and (E, d) is a complete metric space then E is a closed subset of X.

(2) Suppose that $\{p_n\}$ is a sequence in a metric space, X, and $p \in X$. Assuming that every subsequence of $\{p_n\}$ itself has a subsequence which converges to p show that $p_n \to p$.

(3) Give a counterexample to each of the following statements:
(a) Subsets of $\mathbb{R}$ are either open or closed
(b) A closed and bounded subset of a metric space is compact.
(c) In any metric space the complement of a connected set is connected.
(d) The closure of the an open ball $\{x \in X; d(p, x) < r\}$ is the closed ball $\{x \in X; d(p, x) \le r\}$.

(4) Let K_i, $i = 1, \dots, N$, be a finite number of compact sets in a metric space X. Show that $\bigcup_{i=1}^N K_i$ is compact.

(5) Let K be a compact set in a metric space X and suppose $p \in X \setminus K$. Show that there exists a point $q \in K$ such that

$$d(p, q) = \inf\{d(p, x); x \in K\}.$$

Solutions:

(1) (a) Let X be a complete metric space and E a closed subset. Let $\{x_n\}$ be a Cauchy sequence in E. This means $\{x_n\}$ is Cauchy in X too, and thus it has a limit $x \in X$. Since E is closed, $x \in E$. So $\{x_n\}$ is convergent in E.

(b) Let $p \in X$ be a limit point of E. Then there exists a sequence $\{p_n\} \subset E$ which converges to p in X. In particular, $\{p_n\}$ must be Cauchy in X, therefore Cauchy in E. Since E is complete, $\{p_n\}$ must be convergent to some $p' \in E$. By the uniqueness of the limit of $\{p_n\}$ in X, $p = p'$. So $p \in E$.

(2) One criterion of convergence for $\{p_n\}$ is that for every $\epsilon > 0$, we have $d(p_n, p) < \epsilon$ for all but finitely many n's. Assume, by contradiction, that $\{p_n\}$ does not converge to p. Then there exists $\epsilon_0 > 0$ such that the set $\{n \in \mathbb{N} : d(p_n, p) \ge \epsilon_0\}$ is infinite. Note that this set is countable, being an infinite subset of a countable set. Arrange its elements increasingly: $n_1 < n_2 < n_3 < \dots$. We obtain an infinite subsequence $\{p_{n_k}\}$ of the original sequence, such that for all k,

$$d(p_{n_k}, p) \ge \epsilon_0.$$

But then no subsequence of $\{p_{n_k}\}$ can converge to p. (All terms are at distance at least ϵ_0, which is a fixed number, from p.) This is a contradiction.

(3) (a) $[0, 1)$ is neither open, nor closed in $\mathbb{R}$.

(b) One example was in the homework: the closed ball $\overline{B}(0, 1)$ in the space of bounded sequences ℓ^∞, with the supremum metric.

Another example is the subset $E = \{x \in \mathbb{Q} : x > 0, 2 < x^2 < 3\}$ of $\mathbb{Q}$. It is bounded, and with the relative topology, it is also closed (and open) in $\mathbb{Q}$. But it is not compact. For example, there is a decreasing sequence in E which converges to $\sqrt{2}$ in $\mathbb{R}$, and therefore it doesn't have a limit point in $\mathbb{Q}$.

(c) Take the set $\{0\}$ in $\mathbb{R}$. It is compact, bacause it is finite. The complement is $(-\infty, 0) \cup (0, \infty)$, so being a union of two disjoint open sets, it is disconnected.

(d) Take $X = \mathbb{R}$ but with the discrete metric. Then $B(0, 1) = \{x \in \mathbb{R} : d(0, x) < 1\}$ is $B(0, 1) = \{0\}$. This is a finite set, therefore closed. But the closed ball $\{x \in \mathbb{R} : d(0, x) \leq 1\}$ is all of $\mathbb{R}$.

(4) Let $\{G_\alpha\}$ be a collection of open sets which covers $\cup_{i=1}^N K_i$. In particular, it covers every K_i. For every i, we find therefore a finite subcover $\{G_{\alpha,i}\}$ of K_i. Then $\cup_{i=1}^N \{G_{\alpha,i}\}$ must cover $\cup_{i=1}^N K_i$, and it is finite. This is the finite subcover.

(5) First, the set $\{d(p, x) : x \in K\} \subset \mathbb{R}$ is bounded below by 0, and so it has an infimum in $\mathbb{R}$. Let d denote this infimum. By the property of the infimum, there exists a sequence $\{x_n\} \subset K$, such that $d(p, x_n) \to d$, as $n \to \infty$. Since K is compact, $\{x_n\}$ must have a convergent subsequence $\{x_{n_k}\}$. Say the limit of this subsequence is $q \in K$. We claim that $d(p, q) = d$.

Let $\epsilon > 0$ be arbitrary. Since $\{x_{n_k}\}$ converges to q, there exists a N, such that

$$d(x_{n_k}, q) < \frac{\epsilon}{2}, \text{ for all } n_k \geq N.$$

On the other hand, since $\{d(x_n, p)\}$ converges to d, there exists N' such that

$$d(p, x_n) < \frac{\epsilon}{2}, \text{ for all } n \geq N'.$$

Choose k_0, such that $n_{k_0} \geq \max(N, N')$. Then, by the triangle inequality

$$d(p, q) \leq d(p, x_{n_{k_0}}) + d(x_{n_{k_0}}, q) < \epsilon.$$

But $\epsilon > 0$ was arbitrary, therefore $d(p, q) = d$.