18.100BC, SPRING 2007
SOLUTIONS TO PROBLEMS FROM RUDIN

(1) Rudin, Chapter 2, Problem 2

An algebraic equation

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \tag{1}$$

in which at least one of the coefficients a_j is not zero, can have at most n complex solutions. For each integer N consider the set

$$A_N = \big\{ z \in \mathbb{C}; z \text{ satisfies (1) for some integers } a_0, a_1, \dots, a_n, n \\ \text{with } 1 \le |a_0| + |a_1| + \cdots + |a_n| + n \le N \big\}$$

Now, A_N is finite, since there are only finitely many equation here and each has only finitely many solutions. Furthermore, the set of algebraic numbers is

$$A = \bigcup_{N=1}^{\infty} A_N,$$

i.e. every algebraic number is in one of the A_N's. Thus A is a countable union of finite sets, so is countable (it is not finite since the integers are clearly algebraic, as $z = n$ satisfies $z - n = 0$).

(2) Rudin, Chapter 2, Problem 3

If every real number was algebraic then, $\mathbb{R} = A \cap \mathbb{R}$ would be countable. We have shown in class that $\mathbb{R}$ is not countable, so $\mathbb{R} \not\subset A$ and hence there must be a non-algebraic real number; indeed there must be an uncountably infinite set of them.

(3) Rudin, Chapter 2, Problem 4

We have shown in class that the set of rational numbers, $\mathbb{Q} \subset \mathbb{R}$ is countable. Since $\mathbb{R}$ is uncountable it cannot be equal to $\mathbb{Q}$ so there must exist irrational real numbers.