18.100B, SPRING 2004
FINAL EXAM: MAY 20
R. MELROSE

This exam is closed book, no books, papers or recording devices permitted. You may use theorems from class, or the book, provided you can recall them correctly. This includes standard properties of the exponential and trigonometric functions. Remember, the thing I want to see most is clarity! The problems are worth 11 point each, except the last one which is worth 12.

Problem 1

Show that the set $\{z \in \mathbb{C}; z = \cos(e^{it^3+t^2})$ for some $t \in \mathbb{R}\}$ is connected.
Hint: This is the image of the real line under a continuous map into $\mathbb{C}$.

Problem 2

Let X be a compact metric space and let $f : X \longrightarrow Y$ be a map to another metric space Y. Show that f is continuous if and only if $f^{-1}(S) \subset X$ is compact for each closed set $S \subset Y$.

Hint: We know that f is continuous iff $f^{-1}(S) \subset X$ is closed for each $S \subset Y$ which is closed. If X is compact then $f^{-1}(S) \subset X$ is compact if and only if it is closed.

Problem 3

(1) Show that the function $f(x) = \exp(\frac{x^3-x}{x^2+x+1})$ is continuously differentiable on $[0, 1]$.
(2) Prove that there is a point $x_0 \in (0, 1)$ at which $f'(x_0) = 0$.

Hint: Since $x^2+x+1 \geq 1$ for $x \in [0, 1]$, the rational function $(x^3-x)/(x^2+x+1)$ is continuously differentiable on $[0, 1]$ and hence so is f by the chain rule and the differentiability of $\exp$. Clearly $f(0) = 1$ and $f(1) = 1$ so apply the mean value theorem.

Problem 4

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable and suppose that 0 is a local maximum of f, i.e. for some $\epsilon > 0$ $f(x) \leq f(0)$ for all $x \in (-\epsilon, \epsilon)$. Show that $f''(0) \leq 0$.

Hint: Since 0 is a local maximum, $f'(0) = 0$. Since f is twice differentiable f' is continuous. If $f''(0) > 0$ then $f'(x) > 0$ for $0 < x \leq \epsilon$ for some $\epsilon > 0$, since the difference quotient $f'(x)/x$ converges to a $f''(0) > 0$ as $x \to 0$. Then, by the mean value theorem $f(\epsilon) - f(0) = \epsilon f'(x) > 0$ for some $x \in (0, \epsilon)$ so 0 is not a local maximum.

Problem 5

Let $g_n : [0,1] \longrightarrow \mathbb{R}$ be a sequence of differentiable functions such that the sequence $g_n' : [0,1] \longrightarrow \mathbb{R}$ is uniformly bounded.

(1) Give an example of such a sequence which is not pointwise bounded.
(2) Show that if $\int_0^1 g_n dx$ is a bounded sequence in $\mathbb{R}$ then g_n has a uniformly convergent subsequence.

Hint: The sequence of constants $g_n(x) = n$ is an example. The uniform boundedness of the g_n' implies the equicontinuity of the sequence g_n by the mean value theorem – so we only need to show its boundedness to deduce the existence of uniformly convergent subsequence from a standard theorem. So we need to show for instance that $g_n(0)$ is a bounded sequence. By the mean value theorem as before, $g_n(x) - g_n(0)$ is bounded independent of x and n, say by A. If $g_n(0)$ is unbounded, it follows that on some subsequence (without changing notation) either $g_n(0) \geq C_n$ or $g_n(0) \leq -C_n$ where $C_n \to \infty$. Then either then $g_n(x) \geq C_n - A$ or $g_n(x) \leq -C_n + A$ which implies that $\int_0^1 g_n(x)dx$ is unbounded. Thus the sequence must be uniformly bounded and hence there is a uniformly convergent subsequence.

Problem 6

Using standard properties of the cosine function show that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3} \cos(nx)$$

defines a continuously differentiable function on the real line.

Hint: $|\cos(nx)| \leq 1$ so comparison with the convergent series $\sum_{n\geq 1} n^{-3}$ shows this series to be uniformly convergent on the real line. For the term-by-term derivative convergence follows by comparison with n^{-2} so the series of derivatives also converges uniformly, hence by a standard theorem the limit is continuously differentiable.

Problem 7

(1) Explain carefully why the Riemann-Stieltjes integral
$$\int_{-1}^{1} \exp(3x^2) d\alpha$$
exists for any increasing function $\alpha : [-1,1] \longrightarrow \mathbb{R}$.
(2) Evaluate this integral when
$$\alpha = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

Hint. The Riemann-Stieltjes integral exists on $[a,b]$ for any increasing α and any continuous function and $\exp(3x^2)$ is continuous as the composite of continuous functions, namely exp and a polynomial. The continuity of exp follows from the definition as the sum of a power series with radius of convergence $+\infty$. Since α is constant except for the jump of size 1 at 0 the uppoer and lower partial sums for a partition in which 0 is the interior of an interval I are $\sup_I f$ and $\inf_I f$. By continuity these approach the same value, 1 as the length of I decreases. Thus the integral is 1. This also follows from a theorem in Rudin that covers cases where α has only a finite jump.

Problem 8

Let $f : X \longrightarrow \mathbb{R}$ be a continuous function on a compact metric space, X. If $\{x_n\}$ is a sequence in X show that $\{f(x_n)\}$ has a convergent subsequence with limit in $f(X)$.

Hint: Since X is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with limit $x \in X$. Since f is continuous, $f(x_{n_k}) \to f(x)$.

Problem 9

Let $f_n : [0,1] \longrightarrow \mathbb{R}$, $n \in \{0, 1, \dots\}$, be the sequence of functions defined by

$$f_0(x) = 1 \;\forall\; x \in [0,1]$$
$$f_{n+1}(x) = \int_0^x f_n(s)ds \;\forall\; x \in [0,1],\; n \geq 0.$$

(1) Show that each f_n is continuous and evaluate $f_n(0)$.
(2) Show that for each $n \geq 0$ and $x, y \in [0,1]$,
$$|f_{n+1}(x) - f_{n+1}(y)| \leq |x - y| \sup_{s \in [0,1]} |f_n(s)|.$$
(3) Deduce that f_n is a uniformly bounded and equicontinuous family.
(4) Show that $\{f_n\}$ has a convergent subsquence.
(5) Prove that the limit, f, of such a subsequence satisfies
$$f(x) = \int_0^x f(s)ds \;\forall\; x \in [0,1]$$
and hence, or otherwise, deduce that $f(x) = 0$ identically on $[0,1]$.

Hint: The cheapest way is to carry out the integrals, realizing that $f_n(x) = x^n/n!$. Otherwise one can work harder.