18.100B Practice for the final exam

Solutions.

Problems.

1) i) Let $\mathcal{M}$ be a metric space, state the definition of equicontinuity of a subset $E \subseteq C(\mathcal{M}, \mathbb{R})$.
Solution. See Definition 7.22 in Rudin.

ii) Show that if $E \subseteq C(\mathcal{M}, \mathbb{R})$ is compact, then it is equicontinuous. (You may not use the Arzela-Ascoli theorem.) *Typo: Should say that $\mathcal{M}$ is compact.*
Solution. Let $\varepsilon > 0$ be given. Find a finite cover of E by balls of radius ε,

$$E \subseteq B_\varepsilon(f_1) \cup \ldots \cup B_\varepsilon(f_n).$$

Each f_i is uniformly continuous, so there is a $\delta_i > 0$ such that whenever $x, y \in \mathcal{M}$ and $d(x,y) < \delta_i$ we have $|f_i(x) - f_i(y)| < \varepsilon$. Let $\delta = \min\{\delta_1, \ldots, \delta_n\}$, note that if $x, y \in \mathcal{M}$ satisfy $d(x,y) < \delta$ and $f \in E$, then $f \in B_\varepsilon(f_j)$ for some j and

$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| < 3\varepsilon.$$

Hence E is equicontinuous.

2) If $S \subseteq \mathbb{R}^n$, show that the collection of isolated points of S is countable.
Solution. Let $\mathcal{S}$ denote the set of isolated points. For every $s \in \mathcal{S}$ choose a neighborhood $\mathcal{U}(s)$ in $\mathbb{R}^n$ such that $\mathcal{U} \cap S = \{s\}$ and so that $\mathcal{U}(s) \cap \mathcal{U}(t) = \emptyset$ if $s \neq t$. Use that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ to choose a point in each $\mathcal{U}(s)$ with rational coordinates. This defines an injective map $\mathcal{S} \to \mathbb{Q}^n$ and proves that $\mathcal{S}$ is countable.

3) i) Prove that if $\mathcal{M}$ and $\mathcal{N}$ are metric spaces and $g : \mathcal{M} \to \mathcal{N}$ is a uniformly continuous function, then whenever $(x_n) \subseteq \mathcal{M}$ is Cauchy, the sequence $(g(x_n))$ is Cauchy.
Solution. Let $\varepsilon > 0$ find $\delta > 0$ so that $d(x,y) < \delta \implies d(g(x), g(y)) < \varepsilon$, find $N \in \mathbb{N}$ such that $n, m > N \implies d(x_n, x_m) < \delta$ and note that hence, for any $n, m > N$ we have $d(g(x_n), g(x_m)) < \varepsilon$.

ii) Let $\mathcal{M}$ and $\mathcal{N}$ be metric spaces, let $A \subseteq \mathcal{M}$ and let $\overline{A} \subseteq \mathcal{M}$ denote the closure of A. If $\mathcal{N}$ is complete and $h : A \to \mathcal{N}$ is uniformly continuous, prove that there is a unique continuous function $\widetilde{h} : \overline{A} \to \mathcal{N}$ such that $\widetilde{h}(a) = h(a)$ for every $a \in A$.
Solution. For any $a \in \overline{A}$, choose $(a_n) \subset A$ such that $a_n \to a$ and define

$$\boxed{\widetilde{h}(a) = \lim_{n \to \infty} h(a_n)}.$$

This limit exists because (a_n) Cauchy implies $(h(a_n))$ Cauchy and N is complete. Also note that the limit is independent of the choice of sequence (a_n) converging to a; if (b_n) is another sequence in A converging to a then by considering the sequence $a_1, b_1, a_2, b_2, \ldots$ we see that the limits coincide. We need to see that $\widetilde{h}$ is continuous. Let $\varepsilon > 0$ find $\delta > 0$ using uniform continuity so that $a, b \in A$, $d(a,b) < \delta$ implies $d(h(a), h(b)) < \varepsilon$. If $x \in \overline{A}$, $a_n \to x$ so that for some $N \in \mathbb{N}$, $n > N$ implies $d\left(f(x), f(a_n)\right) < \varepsilon$ and we pick any point $b \in B_{\delta/2}(x) \cap A$ then there is a $a_m \in B_{\delta/2}(x)$ with $m > N$ hence

$$|\widetilde{h}(x) - \widetilde{h}(b)| \leq |\widetilde{h}(x) - \widetilde{h}(a_m)| + |\widetilde{h}(a_m) - \widetilde{h}(b)| < 2\varepsilon.$$

Similarly, if $c \in B_{\delta/2}(x) \cap \overline{A}$ then $|\widetilde{h}(x) - \widetilde{h}(c)| < 3\varepsilon$ and hence $\widetilde{h}$ is continuous on $\overline{A}$. Finally, since any continuous function on $\overline{A}$ must satisfy the boxed equation above, the extension is unique.

4) Assume $f : (a, b) \to \mathbb{R}$ has derivative at every point in (a, b). Let $c \in (a, b)$ and assume that

$$\lim_{x \to c} f'(x)$$

exists and is finite. Prove that the value of this limit must be $f'(c)$.
Solution. See Pset 8, question 5.

5) Assume f, g, and h are real-valued functions defined on $[0, 1]$ and $g \geq 0$ is in $\mathcal{R}(x)$.
   i) Prove that if f is continuous, there exists $w \in [0, 1]$ such that

$$\int_0^1 f(t)\, g(t)\ dt = f(w) \int_0^1 g(t)\ dt$$

Hint: Use the intermediate value theorem.
Solution. Notice that

$$\left(\min_{[0,1]} f\right) \int_0^1 g(t)\ dt \leq \int_0^1 f(t)\, g(t)\ dt \leq \left(\max_{[0,1]} f\right) \int_0^1 g(t)\ dt$$

so the problem follows from the intermediate value theorem (using continuity of f).

   ii) Prove that if h is monotone increasing (not necessarily continuous), there exists $z \in [0, 1]$ such that

$$\int_0^1 h(t)\, g(t)\ dt = h(0) \int_0^z g(t)\ dt + h(1) \int_z^1 g(t)\ dt$$

Hint: Use the intermediate value theorem, but make sure to justify continuity.
Solution. Let

$$\Phi(x) = h(0) \int_0^x g(t)\ dt + h(1) \int_x^1 g(t)\ dt,$$

g integrable implies Φ continuous. On the other hand, since $h(0) < h(1)$, $\min \Phi = \Phi(1)$ and $\max \Phi = \Phi(0)$. Finally, since h is monotone increasing

$$\min_{[0,1]} \Phi = h(0) \int_0^1 g(t)\ dt \leq \int_0^1 h(t)\, g(t)\ dt \leq h(1) \int_0^1 g(t)\ dt = \max_{[0,1]} \Phi$$

so the problem follows from the intermediate value theorem applied to Φ.

6) Let $S = \{n_1, n_2, \ldots,\}$ denote the collection of those positive integers that do not involve the digit 3 in their decimal representation. (For example, $7 \in S$, but $131 \notin S$.) Show that $\sum \frac{1}{n_k}$ converges and has sum less than 90.
Hint: If m has ℓ digits, then $\frac{1}{m} \leq \frac{1}{10^{\ell-1}}$. How many elements of S have ℓ digits?
Solution. If $s \in S$ has exactly ℓ digits then the first digit can be anything other than 0 or 3 (8 possibilities) and the other digits can be anything other that 3 (9 possibilities) thus there are $8 \cdot (9)^{\ell-1}$ such numbers. That means that

$$\sum_{n_k \in S} \frac{1}{n_k} < 8 \sum_{\ell \geq 1} \frac{9^{\ell-1}}{10^{\ell-1}} = 8 \sum_{\ell=0}^{\infty} \left(\frac{9}{10}\right)^{\ell} = \frac{8}{1 - \frac{9}{10}} = 80.$$

7) Assume that (g_n) is a sequence of real-valued functions defined on $T \subseteq \mathbb{R}$ satisfying $g_{n+1}(x) \leq g_n(x)$ for each $x \in T$ and $n \in \mathbb{N}$, and suppose that $g_n \to 0$ uniformly on T. Show that

$$\sum_{n=1}^{\infty} (-1)^{n+1} g_n(x)$$

converges uniformly on T.
Solution. Define

$$G_k(x) = \sum_{n=1}^{k} (-1)^{n+1} g_n(x)$$

and note that if $2\ell < k$

$$G_k(x) = G_{2\ell}(x) + (g_{2\ell+1}(x) - g_{2\ell+2}(x)) + (g_{2\ell+3}(x) - g_{2\ell+4}(x)) + \ldots \pm g_k(x) \geq G_{2\ell}(x)$$

and if $2\ell + 1 < k$ then

$$G_k(x) = G_{2\ell+1}(x) - (g_{2\ell+1}(x) - g_{2\ell+2}(x)) - (g_{2\ell+3}(x) - g_{2\ell+4}(x)) - \ldots \pm g_k(x) \leq G_{2\ell+1}(x).$$

Let $\varepsilon > 0$, find $N \in \mathbb{N}$ such that $n > N$ implies $\|g_n\| < \varepsilon$ then if $s, t > 2n+1$ we have

$$G_{2n}(x) \leq G_s(x) \leq G_{2n+1}(x), \qquad \text{and} \qquad G_{2n}(x) \leq G_t(x) \leq G_{2n+1}(x)$$

hence

$$|G_s(x) - G_t(x)| \leq |G_{2n+1}(x) - G_{2n}(x)| = |g_{2n+1}(x)| < \varepsilon$$

which proves that G_s is uniformly Cauchy and hence uniformly convergent.

8) Consider a continuous function $f : [0, \infty) \to \mathbb{R}$. For each n define the continuous function $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = f(x^n)$. Show that the set of continuous functions $\{f_1, f_2, \ldots\}$ is equicontinuous on some interval containing $x = 1$ if and only if f is a constant function.
Solution. Let I be an interval containing 1, let $\varepsilon > 0$ and let $\delta > 0$ be such that $x, y \in I$, $|x - y| < \delta$ implies $f_n(x) - f_n(y)| < \varepsilon$. Thus if $|x - 1| < \delta$ then $|f(1) - f(x^n)| < \varepsilon$ for every $n \in \mathbb{N}$. Choose an $x < 1$ to see that $|f(1) - f(0)| \leq \varepsilon$ and since $\varepsilon > 0$ was arbitrary $f(1) = f(0)$. For any $z \in (0, \infty)$, choose large enough N so that $|z^{\frac{1}{N}} - 1| < \delta$ and hence $|f(1) - f(z)| < \varepsilon$, but since ε was arbitary $f(z) = f(1)$.

9) Define, for any $z \in \mathbb{R}$, the exponential function by

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

i) Prove that $\exp : \mathbb{R} \to \mathbb{R}$ is a continuous function.
Solution. The partial sums are continuous functions that converge uniformly on any compact set (e.g., by the ratio test) and hence the limit function is continuous.

ii) Use the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

to prove $\exp(z+w) = \exp(z)\exp(w)$. Be sure to justify your steps.
Solution. Note that the series converges absolutely.

$$\begin{aligned}\exp(z+w) &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k y^{n-k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{y^{n-k}}{(n-k)!} = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \right) \left(\sum_{\ell=0}^{\infty} \frac{y^\ell}{\ell!} \right) = \exp(z)\exp(w)\end{aligned}$$

The first equality is by definition of exp, the second uses the binomal theorem, the third the definition of $\binom{n}{k}$, the fourth uses absolute convergence to rearrange the summands and the last uses definition of exp again.

iii) Prove that $\exp'(z) = \exp(z)$. Be sure to justify your steps.
Solution. Denote by $S_n(z)$ the partial sums of $\exp z$ and note that $(S_n(z))' = S_{n-1}(z)$ and hence $(S_n(z))'$ converges uniformly. It follows that $\exp z$ is differentiable with derivative

$$\exp'(z) = \lim_n (S_n(z))' = \lim_n S_{n-1}(z) = \exp(z).$$