18.100B Practice for the final exam

Not to be turned in, just for practice.

Problems.

1) i) Let $\mathcal{M}$ be a metric space, state the definition of equicontinuity of a subset $E \subseteq C(\mathcal{M}, \mathbb{R})$.
   ii) Show that if $E \subseteq C(\mathcal{M}, \mathbb{R})$ is compact, then it is equicontinuous. (You may not use the Arzela-Ascoli theorem.)

2) If $S \subseteq \mathbb{R}^n$, show that the collection of isolated points of S is countable.

3) i) Prove that if $\mathcal{M}$ and $\mathcal{N}$ are metric spaces and $g : \mathcal{M} \to \mathcal{N}$ is a uniformly continuous function, then whenever $(x_n) \subseteq \mathcal{M}$ is Cauchy, the sequence $(g(x_n))$ is Cauchy.
   ii) Let $\mathcal{M}$ and $\mathcal{N}$ be metric spaces, let $A \subseteq \mathcal{M}$ and let $\overline{A} \subseteq \mathcal{M}$ denote the closure of A. If $\mathcal{N}$ is complete and $h : A \to \mathcal{N}$ is uniformly continuous, prove that there is a unique continuous function $\widetilde{h} : \overline{A} \to \mathcal{N}$ such that $\widetilde{h}(a) = h(a)$ for every $a \in A$.

4) Assume $f : (a, b) \to \mathbb{R}$ has derivative at every point in (a, b). Let $c \in (a, b)$ and assume that

$$\lim_{x \to c} f'(x)$$

exists and is finite. Prove that the value of this limit must be $f'(c)$.

5) Assume f, g, and h are real-valued functions defined on $[0, 1]$ and $g \geq 0$ is in $\mathcal{R}(x)$.
   i) Prove that if f is continuous, there exists $w \in [0, 1]$ such that

$$\int_0^1 f(t)\, g(t)\ dt = f(w) \int_0^1 g(t)\ dt$$

   Hint: Use the intermediate value theorem.
   ii) Prove that if h is monotone increasing (not necessarily continuous), there exists $z \in [0, 1]$ such that

$$\int_0^1 h(t)\, g(t)\ dt = h(0) \int_0^z g(t)\ dt + h(1) \int_z^1 g(t)\ dt$$

   Hint: Use the intermediate value theorem, but make sure to justify continuity.

6) Let $S = \{n_1, n_2, \ldots,\}$ denote the collection of those positive integers that do not involve the digit 3 in their decimal representation. (For example, $7 \in S$, but $131 \notin S$.) Show that $\sum \frac{1}{n_k}$ converges and has sum less than 90.
Hint: If m has ℓ digits, then $\frac{1}{m} \leq \frac{1}{10^{\ell}}$. How many elements of S have ℓ digits?

7) Assume that (g_n) is a sequence of real-valued functions defined on $T \subseteq \mathbb{R}$ satisfying $g_{n+1}(x) \leq g_n(x)$ for each $x \in T$ and $n \in \mathbb{N}$, and suppose that $g_n \to 0$ uniformly on T. Show that

$$\sum_{n=1}^{\infty} (-1)^{n+1} g_n(x)$$

converges uniformly on T.

8) Consider a continuous function $f : [0, \infty) \to \mathbb{R}$. For each n define the continuous function $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = f(x^n)$. Show that the set of continuous functions $\{f_1, f_2, \ldots\}$ is equicontinuous on some interval containing $x = 1$ if and only if f is a constant function.

9) Define, for any $z \in \mathbb{R}$, the exponential function by

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

i) Prove that $\exp : \mathbb{R} \to \mathbb{R}$ is a continuous function.

ii) Use the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$$

to prove $\exp(z+w) = \exp(z)\exp(w)$. Be sure to justify your steps.

iii) Prove that $\exp'(z) = \exp(z)$. Be sure to justify your steps.