18.100B, FALL 2002
PRACTICE FINAL EXAM: WITH (STRONG) HINTS

Some of these are similar to the questions on the final and some are rejects.

Try each of the questions; the first seven are worth 12 points each, the last one is harder and is worth 16. You may use theorems from class, or the book, provided you can recall them correctly! This includes standard properties of the exponential and trigonometric functions. No books or papers are permitted.

Problem 1

Show that the set $\{z \in \mathbb{C}; 1 < |z| < 2\}$ is connected as a subset of $\mathbb{C}$ with the usual metric.

Hint: Image of a connected set under a continuous map.

Problem 2

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable and satisfy $|g'(x)| \leq \frac{1}{2}$. Show that the function $f(x) = x - g(x)$ is 1-1.

Hint: MVT, $f(x) - f(y) = (x - y)f'(t)$, $f'(t) = 1 - g'(t) \neq 0$ so f is 1-1.

Problem 3

(1) Why is the function $f(x) = |x|^{\frac{5}{2}} \cos(|x|^{\frac{3}{2}})$ continuously differentiable on $[0, 1]$?

(2) Why does f have a minimum value on this interval?

Hint: $|x|^{\frac{3}{2}}$ is differential on $[0, 1]$ – we showed in class that $x^{\frac{1}{2}}$ is differentiable in $x > 0$ so use composition with $|y|^3$ in $y \neq 0$ and check directly at $x = 0$. We know cos is differentiable so $\cos(|x|^{\frac{3}{2}}$ is differentiable as composite of differentiable functions. $|x|^{\frac{5}{2}} = x \times |x|^{\frac{3}{2}}$ is differentiable by product run, as is f. Same for continuous differentiability.

It is a continuous function on a compact metric space.

Problem 4

Let $f[0, 1] \longrightarrow \mathbb{R}$ be continuous. Show that there exists $c \in (0, 1)$ such that $\int_0^1 f(x)dx = f(c)$.

Hint: By FTC $F(x) = \int_0^x f(t)dt$ is differentiable on $[0, 1]$ so by the MVT $F(1) - f(0) = F'(c)$ for some $c \in (0, 1)$.

Problem 5

(1) For what values of $x \in \mathbb{R}$ does the series

$$\sum_{n=0}^{\infty} n \exp(-nx)$$

converge?

(2) For what intervals $[a, b]$ does it converge uniformly?

(3) On what intervals $[a, b]$ is the sum of the series differentiable?

Hint: A necessary condition for convergence is that the terms of a series tend to zero. So certain must have $x > 0$ for convergence. Conversely if $x > 0$ then $n^3 \exp(-xn)$ is bounded, so the series converges.

Terms are positive and decrease as x increases, so it converges on $[a, b]$ if $b \geq a > 0$. It does not converge on $[0, a]$ for any $a > 0$ since by continuity of the terms this would imply convergence at $x = 0$.

The sum of the series is differentiable on $[a, b]$ for any $a > 0$ since the series of derivatives converges for the same reason.

Problem 6

Consider the (power) series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n.$$

Show that this series converges uniformly on $(-\frac{1}{2}, \frac{1}{2})$; let $f(x)$ denote the sum. Show that the series obtained by term-by-term differentiation converges uniformly in the same set and explain why the limit is $f'(x)$. If $f'(x)$ a rational function?

Hint: The radius of convergence is 1, since $\sqrt[n]{n} \longrightarrow 1$. Thus the series converges uniformly on $(-\frac{1}{2}, \frac{1}{2})$. The differentiate power series has all coefficients 1 so converges uniformly for $x \in (-\frac{1}{2}, \frac{1}{2})$ (in fact any interval strictly inside $(-1, 1)$. The uniform convergence of the series of derivatives implies that it converges to $f' = 1/(1 - x)$ which is indeed rational.

Problem 7

(1) Explain carefully why the Riemann-Stieltjes integral

$$\int_0^2 \exp(3(|x|^{\frac{3}{2}} - 1)) d\alpha$$

exists for any increasing function $\alpha : [0, 1] \longrightarrow \mathbb{R}$.

(2) Evaluate this integral when

$$\alpha(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 3 & 1 \leq x \leq 2. \end{cases}$$

Hint: The function $f = \exp(3(|x|^{\frac{3}{2}} - 1))$ is continuous on $[0, 2]$ as the composite of continuous functions and so is integrable with respect to any increasing α.

The given α is constant except for a jump of size 2 at $x = 1$, so the integral reduces to $2f(1) = 2$.

Problem 8

Suppose that $f_n : [0, 1] \longrightarrow \mathbb{R}$ is a sequence of continuous functions which is uniformly bounded and satisfies

$$f_n(x) = \frac{1}{n} + \int_0^x f_n^2(t) dt, \ x \in [0, 1].$$

Show that $\{f_n\}$ is uniformly convergent on $[0, 1]$ and prove that the limit is identically zero.

Hint: The continuity of f_n implies that of f_n^2 so the FTC shows each f_n to be differentiable with derivative $f_n' = f_n^2$. Thus the uniform boundedness of the f_n implies the uniform boundedness of the f_n'. By the MVT this implies that f_n is an equicontinuous sequence of functions. Thus any subsequence of the f_n has a uniformly convergent subsequence. Consider such a uniformly convergent subsequence $f_{n(k)} \to f(x)$ The limit must satisfy $f(x) = \int_0^x f^2(t)dt$. If $T = \sup_{[0,r]} |f(x)|$ then from this equation $T \leq rMT$ which implies that $T = 0$ if $rM^2 < 1$. Repeating the argument shows that $f = 0$ on $[0,1]$. Thus every subsequence of f_n has a uniformly convergent subsequence with limit identically 0. Hence in fact the sequence converges uniformly to 0.