18.100B, SPRING 2004
FINAL EXAM: MAY 20
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This exam is closed book, no books, papers or recording devices permitted. You may use theorems from class, or the book, provided you can recall them correctly. This includes standard properties of the exponential and trigonometric functions. Remember, the thing I want to see most is clarity! The problems are worth 11 point each, except the last one which is worth 12.

Problem 1

Show that the set $\{z \in \mathbb{C}; z = \cos(e^{it^3+t^2})$ for some $t \in \mathbb{R}\}$ is connected.

Problem 2

Let X be a compact metric space and let $f : X \longrightarrow Y$ be a map to another metric space Y. Show that f is continuous if and only if $f^{-1}(S) \subset X$ is compact for each closed set $S \subset Y$.

Problem 3

(1) Show that the function $f(x) = \exp(\frac{x^3-x}{x^2+x+1})$ is continuously differentiable on $[0,1]$.
(2) Prove that there is a point $x_0 \in (0,1)$ at which $f'(x_0) = 0$.

Problem 4

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable and suppose that 0 is a local maximum of f, i.e. for some $\epsilon > 0$ $f(x) \le f(0)$ for all $x \in (-\epsilon, \epsilon)$. Show that $f''(0) \le 0$.

Problem 5

Let $g_n : [0,1] \longrightarrow \mathbb{R}$ be a sequence of differentiable functions such that the sequence $g_n' : [0,1] \longrightarrow \mathbb{R}$ is uniformly bounded.

(1) Give an example of such a sequence which is not pointwise bounded.
(2) Show that if $\int_0^1 g_n dx$ is a bounded sequence in $\mathbb{R}$ then g_n has a uniformly convergent subsequence.

Problem 6

Using standard properties of the cosine function show that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3} \cos(nx)$$

defines a continuously differentiable function on the real line.

Problem 7

(1) Explain carefully why the Riemann-Stieltjes integral

$$\int_{-1}^{1} \exp(3x^2)d\alpha$$

exists for any increasing function $\alpha : [-1,1] \longrightarrow \mathbb{R}$.

(2) Evaluate this integral when

$$\alpha = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

Problem 8

Let $f : X \longrightarrow \mathbb{R}$ be a continuous function on a compact metric space, X. If $\{x_n\}$ is a sequence in X show that $\{f(x_n)\}$ has a convergent subsequence with limit in $f(X)$.

Problem 9

Let $f_n : [0,1] \longrightarrow \mathbb{R}$, $n \in \{0,1,\dots\}$, be the sequence of functions defined by

$$f_0(x) = 1 \ \forall \ x \in [0,1]$$

$$f_{n+1}(x) = \int_0^x f_n(s)ds \ \forall \ x \in [0,1], \ n \geq 0.$$

(1) Show that each f_n is continuous and evaluate $f_n(0)$.

(2) Show that for each $n \geq 0$ and $x, y \in [0,1]$,

$$|f_{n+1}(x) - f_{n+1}(y)| \leq |x-y| \sup_{s\in[0,1]} |f_n(s)|.$$

(3) Deduce that f_n is a uniformly bounded and equicontinuous family.

(4) Show that $\{f_n\}$ has a convergent subsquence.

(5) Prove that the limit, f, of such a subsequence satisfies

$$f(x) = \int_0^x f(s)ds \ \forall \ x \in [0,1]$$

and hence, or otherwise, deduce that $f(x) = 0$ identically on $[0,1]$.