PRACTIVE FOR TEST 2 NUMBER 2
18.100B SPRING 2007

This test is closed book, no books, papers or notes are permitted. You may use theorems, lemmas and propositions from the class and book. Note that where $\mathbb{R}^k$ is mentioned below the standard metric is assumed.

There are 5 questions on the actual test, I think they are mostly easier than these ones.

(1) Consider the function $\alpha : [0,1] \longrightarrow \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} \frac{1}{2}x & 0 \le x \le \frac{1}{2} \\ \frac{1}{2}(x+1) & \frac{1}{2} \le x \le 1. \end{cases}$$

Show carefully, using results from class, that any monotonic increasing function $f : [0,1] \longrightarrow \mathbb{R}$ which is continuous at $x = \frac{1}{2}$ is Riemann-Stieltjes integrable with respect to α.

(2) Let f be a continuous function on $[a,b]$. Explain whether each of the following statements is always true, with brief but precise reasoning.
  - (a) The function $g(x) = \int_x^b f(y)dy$ is well defined.
  - (b) The function g is continuous.
  - (c) The function g is decreasing.
  - (d) The function g is uniformly continuous.
  - (e) The function g is differentiable.
  - (f) The derivative $g' = f$ on $[a,b]$.

(3) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differntiable and satisfies $f(-10) = 10$, $f(0) = 0$, $f(10) = 10$ show that there is a point where $f'(x) = 1/2$.

(4) If f is a strictly positive continuous function on $[-1,1]$, meaning $\inf_{[-1,1]} f > 0$, show that $g(x) = \sqrt{f(x)}$ is continuous.

(5) (This is basically Rudin Problem 4.14)

Let $f : [0,1] \longrightarrow [0,1]$ be continuous.
  - (a) State why the the map $g(x) = f(x) - x$, from $[0,1]$ to $\mathbb{R}$ is continuous.
  - (b) Using this, or otherwise, show that $L = \{x \in [0,1]; f(x) \le x\}$ is closed and $\{x \in [0,1]; f(x) < x\}$ is open.
  - (c) Show that L is not empty.
  - (d) Suppose that $f(x) \ne x$ for all $x \in [0,1]$ and conclude that L is open in $[0,1]$ and that $L \ne [0,1]$.
  - (e) Conclude from this, or otherwise, that there must in fact be a point $x \in [0,1]$ such that $f(x) = x$.

(6) Consider the function

$$f(x) = \frac{-x(x+1)(x-100)}{x^{44} + x^{34} + 1}$$

for $x \in [0,100]$.
  - (a) Explain why f is differentiable.
  - (b) Compute $f'(0)$.

(c) Show that there exists $\epsilon > 0$ such that $f(x) > 0$ for $0 < x < \epsilon$.
(d) Show that there must exist a point x with $f'(x) = 0$ and $0 < x < 100$.