PRACTIVE FOR TEST 2 NUMBER 2 WITH SOLUTIONS
18.100B SPRING 2007

This test is closed book, no books, papers or notes are permitted. You may use theorems, lemmas and propositions from the class and book. Note that where $\mathbb{R}^k$ is mentioned below the standard metric is assumed.

There are 5 questions on the actual test, I think they are mostly easier than these ones.

(1) Consider the function $\alpha : [0,1] \longrightarrow \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} \frac{1}{2}x & 0 \le x \le \frac{1}{2} \\ \frac{1}{2}(x+1) & \frac{1}{2} \le x \le 1. \end{cases}$$

Show carefully, using results from class, that any monotonic increasing function $f : [0,1] \longrightarrow \mathbb{R}$ which is continuous at $x = \frac{1}{2}$ is Riemann-Stieltjes integrable with respect to α.

Solution: Write $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = \frac{1}{2}x$ and $\alpha_2 = 0$ in $x \le \frac{1}{2}2$, $\alpha_2 = \frac{1}{2}$ in $\frac{1}{2} < x \le 1$. Then a_1 is continuous and as f is monotonic, $f \in \mathcal{R}(\alpha_1)$ by a result in the book. Since $\alpha_2 = \frac{1}{2}I(x - \frac{1}{2})$ and f is continuous at $\frac{1}{2}$ combinging two results from the books shows that $f \in \mathcal{R}(\alpha_2)$. From this it follows that $f \in \mathcal{R}(\alpha)$.

(2) Let f be a continuous function on $[a,b]$. Explain whether each of the following statements is always true, with brief but precise reasoning.
   (a) The function $g(x) = \int_x^b f(y)dy$ is well defined.
   Yes, $f \in \mathcal{R}$ for any subinterval.
   (b) The function g is continuous.
   Yes, the integral is a continous function of the lower limit.
   (c) The function g is decreasing.
   No, not unless $f \ge 0$.
   (d) The function g is uniformly continuous.
   Yes, continous on a compact set implies uniformly continuous.
   (e) The function g is differentiable.
   Yes, g is differentiable since f is continuous.
   (f) The derivative $g' = f$ on $[a,b]$.
   No, you fiend, it is $g' = -f$ since it is the lower limit!

(3) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differntiable and satisfies $f(-10) = 10$, $f(0) = 0$, $f(10) = 10$ show that there is a point where $f'(x) = 1/2$.

Solution: Applying the mean value theorem twice, there are points $z \in (-10,0)$ where $f'(z) = -1$ and $y \in (0,10)$ where $f(y) = 1$. From the intermediate value theorem for derivatives there must exist a point $x \in (z,y)$ at which $f'(x) = \frac{1}{2}$.

(4) If f is a strictly positive continuous function on $[-1,1]$, meaning $\inf_{[-1,1]} f > 0$, show that $g(x) = \sqrt{f(x)}$ is continuous.

Solution: The function $\sqrt{\cdot}:(0,\infty) \longrightarrow (0,\infty)$ is continuous since if $x, y > t > 0$, $t < 1$, then

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} < \epsilon$$

if $|x - y| \leq \epsilon t$ (since $\sqrt{t} > t$). The composite of two continuous functions is continuous so $\sqrt{f} : [-1, 1] \longrightarrow (0,\infty)$ is continuous.

(5) (This is basically Rudin Problem 4.14)

Let $f : [0, 1] \longrightarrow [0, 1]$ be continuous.

(a) State why the the map $g(x) = f(x) - x$, from $[0, 1]$ to $\mathbb{R}$ is continuous.
(b) Using this, or otherwise, show that $L = \{x \in [0, 1]; f(x) \leq x\}$ is closed and $\{x \in [0, 1]; f(x) < x\}$ is open.
(c) Show that L is not empty.
(d) Suppose that $f(x) \neq x$ for all $x \in [0, 1]$ and conclude that L is open in $[0, 1]$ and that $L \neq [0, 1]$.
(e) Conclude from this, or otherwise, that there must in fact be a point $x \in [0, 1]$ such that $f(x) = x$.

Solution:

(a) $g(x) = -x$ is continous and the sum of two continuous functions is continuous.
(b) $L = \{x \in [0, 1]; f(x) \leq x\} = g^{-1}((-\infty, 0])$ is the inverse image of a closed set under a continous map, so is closed. Similarly, $\{x \in [0, 1]; f(x) < x\} = g^{-1}((-\infty, 0))$ is the inverse image of an open set under a continous map, so is open.
(c) Since $g(1) = f(1) - 1 \leq 0$, $1 \in L$ so $L \neq \emptyset$.
(d) If $f(x) \neq x$ for all $x \in [0, 1]$ then $L = g^{-1}((-\infty, 0])$ since $g^{-1}(\{0\}) = \emptyset$. Thus, L is open in $[0, 1]$. However, $L \neq [0, 1]$ since $g(0) \geq 0$ so $0 \notin L$.
(e) The preceeding statements are not consistent, that $L \subset [0, 1]$ is both open and closed, non-empty and not equal to $[0, 1]$ since $[0, 1]$ is connected. Thus the assumption $f(x) \neq x$ for all $x \in [0, 1]$ is incorrect and there must exist some $x \in [0, 1]$ such that $f(x) = x$.

(6) Consider the function

$$f(x) = \frac{-x(x+1)(x-100)}{x^{44} + x^{34} + 1}$$

for $x \in [0, 100]$.

(a) Polynomials are differentiable and $x^{44} + x^{34} + 1 \neq 0$ on the real line, so f is differentiable as the quotient of a differentiable function by a non-vanishing differentiable function.
(b) $f'(0) = 100$.
(c) The definition of differentiability means that there exists $\epsilon > 0$ such that the difference quotient $(f(x) - f(0))/x > 0$ for $0 < x < \epsilon$ since its limit at $x = 0$ is 100. Thus $f(x) > 0$ for $0 < x < \epsilon$.
(d) By inspection, $f(0) = f(100) = 0$. The mean value theorem asserts the existence of $x \in (0, 100)$ such that $f(100) - f(0) = 100f'(x)$, so $f'(x) = 0$.