18.100B TEST 2, 27 APRIL 2004
11:05AM – 12:25PM

This test is closed book, no books, papers or notes are permitted. You may use theorems, lemmas and propositions from the class and book. Note that where $\mathbb{R}^k$ is mentioned below the standard metric is assumed.

Problem 1 Put $B = \{(x, y) \in \mathbb{R}^2; x^2+y^2 \le 1\}$ and suppose $g : B \longrightarrow \mathbb{R}$ is a continuous function. Show that the function

$$f(x, y) = x^2 + y^2 + (x^2 + y^2)(1 - (x^2 + y^2))g(x, y)$$

is continuous on B and prove that there is some point $(\bar{x}, \bar{y}) \in B$ such that $f(\bar{x}, \bar{y}) = \frac{1}{2}$.

Solution: polynomial are continuous, sums of continuous functions are continuous and products of continuous functions are continuous, so f is continuous on B. Now, B is connected (being a convex subset of $\mathbb{R}^2$) so the image $f(B) \subset \mathbb{R}$ is connected. On the other hand we can see that $f(0, 0) = 0$ and $f(1, 0) = 1$ so $[0, 1] \subset f(B)$ since a connected subset of $\mathbb{R}$

contains the interval between and two points in it. Thus $\frac{1}{2} \in f(B)$ so there exists $(x, y) \in B$ with $f(x, y) = \frac{1}{2}$.

Problem 2 If $f : [0, 1] \longrightarrow [0, \infty)$ is increasing and $f(\frac{1}{2}) > 1$, show that $\int_0^1 f(x)dx > \frac{1}{2}$.

Solution: Since f is increasing and $f(\frac{1}{2}) > 1$, $f(x) > 1$ for all $x \in [\frac{1}{2},]$. As an increasing function, f is Riemann integrable and

$$\int_0^1 f(x)dx \geq \int \frac{1}{2}^1 f(x)dx \geq \frac{1}{2} \inf_{[\frac{1}{2},1]} f > \frac{1}{2}. \tag{1}$$

Problem 3 Show that if f is a continuous function on $[a, b]$ then there exists a function $g : [a, b] \longrightarrow \mathbb{R}$ such that $g' = f$.

Solution. Since f is continuous, the fundamental theorem of calculus, shows that $g(x) = \int_a^x f(t)dt$ is differentiable and satisfies $g'(x) = f(x)$.

Problem 4 If $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are two functions which are continuous at 0, show that the function

$$h(x) = \max\{f(x), g(x)\},\ x \in \mathbb{R}$$

is also continuous at 0.

Solution: The continuity of both f and g implies that given $\epsilon > 0$ there exists $\delta > 0$ sucht that for $|x| < \delta$, $|f(x) - f(0)| < \epsilon$ and $|g(x) - g(0)| < \epsilon$. If $f(0) > g(0)$ it follows, by choosing $\epsilon < \frac{1}{2}(f(0) - g(0))$ that $f(x) > g(x)$ in $|x| < \delta$, and hence that $h(x) = f(x)$ in $|x| < \delta$ is continuous. If $g(0) > f(0)$ the same argument with f and g exchanged shows that h is continuous at

0. Finally, if $f(0) = g(0) = h(0)$ then the choice of δ above shows that

$$h(0) - \epsilon < f(x) < h(0) + \epsilon, h(0) - \epsilon < g(x) < h(0) + \epsilon \tag{2}$$

so the same is true for $h(x) = \max(f(x), g(x))$ which is therefore continuous at 0.

Problem 5 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable functions which satisfies

$$f(x) = 0, \text{ for } |x| > 10, \ f(-1) = 1, \ f(1) = -1.$$

Show that there are at least two values of $x \in \mathbb{R}$ such that $f'(x) = 0$.

Solution: Consider the max and min of f on $[-10, 10]$. Since the max is at least 1 and the min at least -1 these values are assumed at points in $(-10, 10)$ and $f'(x)$ must vanish at both.