18.100B Practice for the first midterm

Solutions.

Problems.

1) Let $(\mathcal{M}, d)$ be an arbitrary metric space.
 a) State the definition of a connected subset of $\mathcal{M}$.

 Solution. See Definition 2.45 in Rudin.

 b) Prove that $E \subseteq M$ is connected if and only if every non-empty proper subset has a non-empty boundary in E.
 Solution. Notice that the equivalent statement
 E is separated if and only if there is a proper non-empty subset with empty boundary in E,
 follows from the fact that $A \cup B$ is a separation of E if and only if A and $B = E \cap A^c$ have no boundary in E.

2) Let $(\mathcal{M}, d)$ be an arbitrary metric space (e.g., not necessarily Euclidean space).
 a) Show that a compact subset of $\mathcal{M}$ is necessarily closed and bounded.
 Solution. See Theorem 2.34 in Rudin for a proof that compact sets are closed. To prove that a compact set $K \subseteq \mathcal{M}$ is bounded, pick any point $p \in \mathcal{M}$ and consider the open sets $B_n(p)$. These cover K (indeed, they cover $\mathcal{M}$), hence there is a finite subcover of K and hence K is contained in $B_N(p)$ for large enough N, i.e., K is bounded.

 b) Give an example of a metric space with a closed and bounded subset that is *NOT* compact.
 Hint: Use the discrete metric $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$
 Solution. Notice that any subset of a metric space with the discrete metric is closed and bounded. However, only finite subsets are compact (by a homework question), hence any infinite subset is closed, bounded, and not compact.

3) Show that $\sqrt{2} + \sqrt{3}$ is irrational.
Hint: Show that $\sqrt{2} + \sqrt{3} \in \mathbb{Q} \implies \sqrt{2} \in \mathbb{Q}$.
Solution. Let $\sqrt{2} + \sqrt{3} = r$ then $\sqrt{3} = r - \sqrt{2}$ and squaring both sides we get $3 = r^2 - 2\sqrt{2} + 2$. If r is rational, then solving this equation for $\sqrt{2}$ would give a rational expression for $\sqrt{2}$ which we know does not exist.

4) Let $(\mathcal{M}, d)$ be an arbitrary metric space (e.g., $\mathcal{M}$ is not necessarily complete). If (x_n) and (y_n) are both Cauchy sequences and $d_n = d(x_n, y_n)$, show that (d_n) is a convergent sequence of real numbers.
Solution. Because $\mathbb{R}$ is complete, we only need to show that d_n is Cauchy. Repeated use of the

triangle inequality shows that

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \implies d(x_n, y_n) - d(x_m, y_m) \le d(x_n, x_m) + d(y_n, y_m)$$

and since the same is true reversing the roles of m and n, we find

$$|d_n - d_m| = |d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_n, y_m).$$

Thus (x_n) Cauchy and (y_n) Cauchy together imply (d_n) Cauchy and hence convergent.

5) Let $(\mathcal{M}, d)$ be an arbitrary metric space. If $G \subseteq \mathcal{M}$ is open, and A is any subset of $\mathcal{M}$, show that

$$G \cap A = \emptyset \iff G \cap \overline{A} = \emptyset$$

Solution. Clearly $G \cap \overline{A} = \emptyset$ implies $G \cap A = \emptyset$, so suppose $G \cap A = \emptyset$, we need to show that no point of G is a limit point of A. But if $x \in G$ then, because G is open, there is an open ball around x that stays in G and hence does not intersect A, which implies that x is not a limit point of A.

6) Show that if $x, y \in \mathbb{R}$ and $x < y$ then there exists an *irrational* number between x and y. (You may use the existence of a rational number between x and y.)
Solution. Let r be a rational number satisfying $x < r < y$, we can find a large enough N so that

$$x < r + \frac{\sqrt{2}}{N} < y \qquad \left(\iff N(y - r) > \sqrt{2}\right)$$

and notice that if $r + \frac{\sqrt{2}}{N} = q$ were rational then $\sqrt{2} = N(q - r)$ would be rational.

An alternate proof is to note that there are uncountably many reals between x and y and there are only countably many rationals, so there must be irrationals between x and y.