18.100 Midterm 1 Solutions

(1) (10 points)

a) Write down the definition of compactness in an arbitrary metric space (not necessarily Euclidean).
Solution. See Definition 2.32 in Rudin.

b) Why are finite sets always compact?
Solution. Given an open cover of a finite set, E, choose for each point in E an open set from the cover containing that point. The resulting collection of open sets is a finite subcover.

(2) (10 points)

Let $K \subseteq X$ be a compact subset of a metric space X. Show that every closed subset $E \subseteq K$ is compact.
Solution. See Theorem 2.35 in Rudin.

(3) (10 points)

Let (p_n) be a Cauchy sequence in an arbitrary metric space X (not necesarrily Euclidean). Prove the following statement: If (p_n) has a convergent subsequence, then the sequence (p_n) itself converges.
Solution. Let (p_{n_k}) be the convergent subsequence, and let p be its limit. We show that $p_n \to p$. Fix $\varepsilon > 0$.
Because p_n is Cauchy, there is an $L \in \mathbb{N}$ such that $s, t > L$ implies $d(p_s, p_t) < \varepsilon$.
Because p_{n_k} converges to p, there is an $M \in \mathbb{N}$ such that $n_k > M$ implies $d(p_{n_k}, p) < \varepsilon$.
Let $N = \max L, M$. Notice that if $s > N$ then

$$d(p_s, p) \leq d(p_s, p_{n_\ell}) + d(p_{n_\ell}, p) < 2\varepsilon$$

where we choose any ℓ so that $n_\ell > N$. Since ε was arbitrary this implies $p_n \to p$.

(4) (10 points)

Suppose $E \subseteq X$ is a nonempty subset of an arbitrary metric space X with metric $d(x, y)$. Define the function $d_E : X \to \mathbb{R}$ by

$$d_E(x) = \inf_{y \in E} d(x, y)$$

Show that the closure of E is given by $\overline{E} = \{y \in X : d_E(y) = 0\}$.
Solution. Notice that $\inf_{y \in E} d(x, y) = 0$ is the same as saying that, for any $\varepsilon > 0$ there exists a $y \in E$ such that $d(x, y) < \varepsilon$. So if $d_E(x) = 0$ and $x \notin E$ then x is a limit point of E and hence in $\overline{E}$. And if x is in E or is a limit point of E then $d_E(x) = 0$.

(5) (10 points)

In what follows all sets are supposed to be subsets of the metric space $\mathbb{R}$ with its usual Euclidean metric $d(x, y) = |x - y|$.

a) Give an example of an infinite collection of closed sets $\{F_n\}_{n=1}^{\infty}$ such that its union $\bigcup_{n=1}^{\infty} F_n$ is not closed.

Solution. Notice that the closed sets $F_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$ satisfy $\bigcup_{n=1}^{\infty} F_n = (-1, 1)$ which is not closed.

b) Given an example of a set that is both open and closed.
Solution. The only right answers are $\emptyset$ or $\mathbb{R}$

c) Given an example of a set that is neither open nor closed.
Solution. For instance, $(0, 1]$.

d) Construct a set E containing a point that is not a limit point of E.
Solution. The set $\{0\}$ containing only the point zero, or the set $[0, 1] \cup \{2\}$ are both examples of correct answers.

e) Construct a set E so that *every* point in E is a limit point of E.
Solution. The set $[0, 1]$ and the Cantor set are both examples of correct answers.