HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 SOLUTIONS TO ASSIGNMENT 8: DUE APRIL 12, AT 11:00 IN 2-108.

(1) (10 points) Let K_1, $K_2 \subset M$ be two a compact subsets of a metric space (M, d). Show that there exist points $p \in K_1$ and $q \in K_2$ such that

$$d(p, q) = \sup_{y \in K_2} \inf_{x \in K_1} d(x, y).$$

Define

$$D(K_1, K_2) = \max \left(\sup_{y \in K_2} \inf_{x \in K_1} d(x, y), \sup_{x \in K_1} \inf_{y \in K_2} d(x, y) \right).$$

Show that D defines a metric on the collection of (non-empty) compact subsets of M.

(RBM) Since I messed this up initially, here is a hint:- The triangle inequality is the tricky part of course. For three compact sets K_1, K_2 and K_3 write down the usual triangle inequality for $x \in K_1$, $y \in K_2$ and $z \in K_3$. Take the infimum over y and then bound one term by $D(K_3, K_2)$ and then take the infimum over z and the supremum over x.

Solution:- First fix y in K_2 and consider $\gamma(y) = \inf_{x \in K_1} d(x, y)$. By definition there is a sequence x_n in K_1 such that $d(x_n, y) \longrightarrow \gamma(y)$. By the compactness of K_1 we may pass to a subsequence which converges to $x'(y)$ and then by the continuity of d, $d(x'(y), y) = \gamma(y)$. Now take the supremum in y so again there exists a sequence $y_n \in K_2$ such that $\gamma(y_n) \to \sup_{y \in K_2} \inf_{x \in K_1} d(x, y)$. Using the compactness of K_2 there is a subsequence which converges $y_n \to y$. The corresponding sequence in K_1, $x'(y_n)$ then has a subsequence which converges to some x, so passing to the subsequence of y_n to which this corresponds we can arrange that $x'(y_n) \to x$ and $y_n \to y$. Now, again by the continuity of the distance function,

$$d(x, y) = \lim d(x'(y_n), y_n) = \sup_{y \in K_2} \inf_{x \in K_1} d(x, y)$$

To see that $D(K_1, K_2)$ defines a metric on compact subsets of M observe first that if $\sup_{y \in K_2} \inf_{x \in K_1} d(x, y) = 0$ then $\inf_{x \in K_1} d(x, y) = 0$ for each $y \in K_2$ and this implies that $d(x'(y), y)$ in the notation above, so $x'(y) = y$ and $K_2 \subset K_1$. It follows that $D(K_1, K_2) = 0$ implies that $K_1 \subset K_2$ and $K_2 \subset K_1$ so $K_1 = K_2$. The converse is obvious so the first condition on a metric is satisfied. Symmetry of D is immediate from the definition. For the triangle inequality, following the hint above, start from

$$d(x, y) \leq d(x, z) + d(z, y) \; \forall \; x \in K_1, \;\; y \in K_2 \text{ and } z \in K_3.$$

Taking the infimum over y,

$$\begin{aligned} \inf_{y \in K_2} d(x, y) &\leq d(x, z) + \inf_{y \in K_2} d(z, y) \\ &\leq d(x, z) + D(K_3, K_2), \;\; \forall \; x \in K_1, \;\; z \in K_2, \end{aligned}$$

since $\sup_{z\in K_3}\inf_{y\in K_2} d(z,y) \le D(K_3,K_2)$. Now take the infimum over z

$$\inf_{y\in K_2} d(x,y) \le \inf_{z\in K_3} d(x,z) + D(K_3,K_2),\ \forall\, x\in K_1.$$

Then take the supremum over $x\in K_1$, to see that

$$\sup_{x\in K_1}\inf_{y\in K_2} d(x,y) \le \sup_{x\in K_1}\inf_{z\in K_3} d(x,z) + D(K_3,K_2) \le D(K_1,K_3) + D(K_3,K_2).$$

Either by repeating the argument with the roles of x and y reversed, or by noting that the right side is symmetric in K_1 and K_2 the inequality

$$\sup_{y\in K_2}\inf_{x\in K_1} d(x,y) \le \sup_{x\in K_1}\inf_{z\in K_3} d(x,z) + D(K_3,K_2) \le D(K_1,K_3) + D(K_3,K_2).$$

also holds and hence

$$D(K_1,K_2) \le D(K_1,K_3) + D(K_3,K_2).$$

This D is a metric as claimed.

(2) (10 points) If $f:[a,b]\longrightarrow\mathbb{R}$ is differentiable (where $a<b$) and $f'(x)\neq 0$ for all $x\in(a,b)$ show that $f(b)\neq f(a)$.

Solution:- By the mean value theorem there exists $x\in(a,b)$ such that

$$f(b)-f(a) = f'(x)(b-a).$$

So if $f(b)=f(a)$ the right side must vanish and since $b>a$ there must exists a point where $f'(x)=0$ violating the assumptions, so $f(b)\neq f(a)$.

(3) (10 points) If C_i for $0\le i\le n$ are real constants such that

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

show that the equation

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n = 0$$

has at least one real solution x in the interval $(0,1)$.

Solution:- Consider the polynomial

$$p(t) = \sum_{i=0}^{n} \frac{C_i}{i+1} t^{i+1}.$$

Since it has no constant term, $p(0)=0$. On the other hand the assumption above is that $p(1)=0$. So, by the mean value theorem there exists some point $x\in(0,1)$ such that $p'(x)=0$. However

$$p'(x) = \sum_{i=0}^{n} C_i x^i$$

so this gives a solution of the equation as desired.

(4) (10 points) Suppose $f:\mathbb{R}\longrightarrow\mathbb{R}$ is differentiable and that $f'(x)\neq 1$, show that there can be *at most one* $x\in\mathbb{R}$ such that $f(x)=x$ ('a fixed point of f').

Solution:- Suppose to the contrary that there are two distinct points $x_1<x_2$ with the property $f(x_i)=x_i$, that is $g(x_1)=g(x_2)=0$ where $g(x)=f(x)-x$. Since g is differentiable, by the mean value theorem there is a point $x\in(x_1,x_2)$ at which $g'(x)(x_2-x_1)=0$, so $g'(x)=f'(x)-1=0$. Hence there can be at most one point at which $f(x)=x$.

(5) (10 points) A function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be 'Lipschitz continuous' (or just 'Lipschitz') if there exists a constant A such that

$$|f(x) - f(y)| \leq A|x - y| \ \forall \ x, y \in [a, b].$$

Show that if $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable and $f' : [a, b] \longrightarrow \mathbb{R}$ is bounded then f is Lipschitz.

Solution:- The Lipschitz constant can be taken to be $A = \sup_{z\in[a,b]} |f'(z)|$ since for any $x \neq y$ there exists z such that

$$f(x) - f(y) = f'(x)(x - y) \Longrightarrow |f(x) - f(y)| \leq A|x - y|.$$

Hence f is Lipschitz, the case $x = y$ being trivially true.

(6) (10 points) Suppose that $g : [0, 1] \longrightarrow \mathbb{R}$ is a Lipschitz function and that $f : [0, 1] \longrightarrow [0, 1]$ is a differentiable functions satisfying

$$f'(x) = g(f(x)) \ \forall \ x \in [0, 1].$$

Show that $f' : [0, 1] \longrightarrow \mathbb{R}$ is Lipschitz.

Solution:- Since f is differentiable, it is continuous. Since g is Lipschitz, it is continuous on an interval containing the range of f, so $g \circ f$ is also continuous. Hence from the differential equation, f' is continous on $[0, 1]$. As a continous function on a compact interval it is bounded, hence by the previous problem f is Lipschitz. Since g is assumed to be Lipschitz, $|g(z) - g(z')| \leq B|z - z'|$ for some constant B and hence

$$|f'(x) - f'(y)| = |g(f(x)) - g(f(y))| \leq B|f(x) - f(y)| \leq AB|x - y|$$

where A is the Lipschitz constant for f.

* Extra Problem – for your amusement only:- Rudin problem 15 of Chapter 5.