HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007
ASSIGNMENT 7: Solutions.

1. Let n be an arbitrary positive integer. Using the definitions, prove that the function $f : [-1, 1] \to \mathbb{R}$, $f(x) = x^n$ is uniformly continuous.

Solution. For every $x, y \in [-1, 1]$, we have $|f(x) - f(y)| = |x^n - y^n| = |x - y| \cdot |x^{n-1} + x^{n-2}y + \cdots + y^{n-1}|$. Since $|x^{n-1} + x^{n-2}y + \cdots + y^{n-1}| \leq |x|^{n-1} + |x|^{n-2}|y| + \cdots + |y|^{n-1} \leq n$, for all $x, y \in [-1, 1]$, we find that

$$|f(x) - f(y)| \leq n|x - y|, \text{ for all } x, y \in [-1, 1].$$

Let $\epsilon > 0$ be given. Set $\delta = \epsilon/n$. For every x, y in $[-1, 1]$, such that $|x-y| < \delta$, it follows that $|f(x) - f(y)| < \epsilon$.

In conclusion, $f(x)$ is uniformly continuous.

2. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that f has a fixed point, i.e., there exists $x \in [0, 1]$, such that $f(x) = x$.

Solution. Consider the function $g(x) : [0, 1] \to \mathbb{R}$, $g(x) = f(x) - x$. Since $f([0, 1]) \subset [0, 1]$, $g(0) = f(0) \geq 0$, and $g(1) = f(1) - 1 \leq 0$. So $g(1) \leq 0 \leq g(0)$. The function $g(x)$ is continuous (because $f(x)$ is), so by the intermediate value property, there exists x, $0 \leq x \leq 1$, such that $g(x) = 0$ (equivalently $f(x) = x$).

3. Let $f, g : X \to Y$ be continuous functions, and let E be a dense subset of X.

(1) Prove that $f(E)$ is dense in $f(X)$.

(2) If $f(p) = g(p)$ for all $p \in E$, prove that $f(x) = g(x)$ for all $x \in X$.

(In other words, this exercise shows that a continuous function is determined by its values on a dense subset of its domain.)

Solution. (a) We prove that every open subset of $f(X)$ intersects $f(E)$. Let V be an open subset of $f(X)$. There exists V' open in Y, such that $V' \cap f(X) = V$. Since $f : X \to Y$ is continuous, $f^{-1}(V')$ is open in X. Then $f^{-1}(V') \cap E \neq \emptyset$, because E is dense in X. This implies that $f(f^{-1}(V')) \cap f(E) \neq \emptyset$. But $f(f^{-1}(V')) \subset V'$, and so $V \cap f(E) = V' \cap f(E) \neq \emptyset$.

(b) Let $x \in X$ be arbitrary. From the density of E, we know that there exists a sequence $\{p_n\} \subset E$, such that $\lim_{n\to\infty} p_n = x$. For every n, $f(p_n) = g(p_n)$, and by the continuity of f and g, it follows that $f(x) = g(x)$.

4. Recall that every rational number $x \in \mathbb{Q}$, $x \neq 0$, can be written uniquely in a reduced form $x = \frac{m}{n}$, where $m, n \in \mathbb{Z}$, $n > 0$. For $x = 0$, take $n = 1$. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q}, \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q}. \end{cases}$$

(1) Prove that f is continuous at every irrational point.
(2) Prove that f is discontinuous at every rational point.

Solution. Let x_0 be an irrational number. Let $\epsilon > 0$, we want to find $\delta > 0$, such that $|f(x) - 0| < \epsilon$, whenever $|x - x_0| < \delta$. Note that if x is irrational, then $|f(x)| = 0$, so the question is how to choose δ so that the inequality holds for rational $x = \frac{m}{n}$ as well (in that case, $|f(x)| = \frac{1}{n}$). Intuitively, if $\frac{m}{n}$ is close to x_0 irrational, then n must be large, but we need to formalize this.

Let N be a natural number, such that $N > \frac{1}{\epsilon}$. If $x = \frac{m}{n}$, $|x - x_0| = \frac{|m-nx_0|}{n}$. For every $i = 1, \ldots, N$, define k_i to be the minimum of the set $\{|m - ix_0| : m \in \mathbb{N}\}$. Note that although this set is infinite, the minimum exists (it comes from the fact that $\mathbb{N}$ is discrete). Moreover, since x_0 is irrational, $k_i > 0$.

Now let $k > 0$ be the minimum of all k_i (here the minimum exists because there are finitely many i's). Set $\delta = \frac{k}{N}$.

The claim is that for all $m, n \in \mathbb{N}$, such that $\frac{|m-nx_0|}{n} < \delta$, necessarily $n > N$. Assume that $n \leq N$. By the construction above $|m - nx_0| \geq k$, so $\frac{|m-nx_0|}{n} \geq \frac{|m-nx_0|}{N} \geq \frac{k}{N} = \delta$, contradiction. So $n \geq N$.

Since $n \geq N$, $n \geq \frac{1}{\epsilon}$, so $f(\frac{m}{n}) < \epsilon$, for all m, n such that $|\frac{m}{n} - x_0| < \delta$. This finishes the proof that f is continuous at irrational x_0.

Now let x_0' be a rational number. By the same argument as before, one can show that $\lim_{x \to x_0'} f(x) = 0$. (The argument is identical, because we only consider, as we should, rationals $\frac{m}{n} \neq x_0'$, which makes all $k_i > 0$ again.)

But since $f(x_0') \neq 0$, it follows that f is discontinuous at x_0', and the discontinuity is of the first kind (the one sided limits exist).

5. Let X be a compact subset of $\mathbb{R}$ and $f : X \to \mathbb{R}$ be a function. Define the *graph* of f to be the set

$$\mathcal{G}(f) = \{(x, f(x)) : x \in X\}.$$

Prove that f is continuous on X if and only if $\mathcal{G}(f)$ is compact.

Solution. Assume first the $\mathcal{G}(f)$ is compact. Assume, by contradiction, that f is discontinuous at $x_0 \in X$. This means that there exists $\epsilon > 0$, and a sequence $\{x_n\} \subset X$, such that $\{x_n\}$ converges to x_0 (for example, $d(x_n, x_0) < \frac{1}{n}$) but $d(f(x_n), f(x_0)) \geq \epsilon$, for all n. Consider the sequence $\{(x_n, f(x_n))\} \subset \mathcal{G}(f)$. Since $\mathcal{G}(f)$ is compact, this must have subsequence $\{(x_{n_k}, f(x_{n_k}))\}$ which converges to a point $(x_1, f(x_1)) \in \mathcal{G}(f)$. In particular, $\{x_{n_k}\}$ converges to x_1. By uniqueness of the limit, $x_1 = x_0$. But then $\{f(x_{n_k})\}$ converges to $f(x_0)$, which is a contradiction with the construction of $\{x_n\}$.

Note that apparently we didn't use that X is compact in this proof. But this is in fact a necessary consequence of the fact that $\mathcal{G}(f)$ is compact. This is because the first projection function $pr_1 : \mathcal{G}(f) \to X$, $(x, f(x)) \mapsto x$, is continuous and surjective.

Conversely, let's assume first that f is continuous. Then $f(X)$ is compact. Consider the set $E = X \times f(X) = \{(x, y) : x \in X, y \in f(X)\}$. Clearly $\mathcal{G}(f) \subset E$. Moreover, $\mathcal{G}(f)$ is closed in E : let $(x_0, y_0) \in E$ be a limit point of $\mathcal{G}(f)$.

There exists a sequence $\{(x_n, f(x_n)\} \subset \mathcal{G}(f)$ which converges to (x_0, y_0). This means that $\{x_n\}$ converges to x_0, and $\{f(x_n)\}$ converges to y_0. Since f is continuous, it must be that $y_0 = f(x_0)$, equivalently $(x_0, y_0) \in \mathcal{G}(f)$.

We would like to claim that E is compact, and therefore $\mathcal{G}(f)$ is compact, being closed in E. If E is a subset of the Euclidean $\mathbb{R}^2$, then it is easy to verify the Heine-Borel theorem for E. The boundedness is immediate: if X and $f(X)$ are bounded, then each is a subset of a 1-cell, therefore E is a subset of some 2-cell. If (x, y) is a limit point of E in $\mathbb{R}^2$, then x is a limit point for X, therefore $x \in X$, and similarly y is a limit point of $f(X)$, so $y \in f(X)$.

Extra problem: Continuous extensions: ex 13 pp 99-100.

Solution. The solution will be posted on Thursday because of the CI-M assignment this week.