HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 7: DUE THURSDAY, APRIL 5, AT 11:00 IN 2-108.

Please remember to tell us which lecture section you are in (Ciubotaru, Melrose, or Parker).

(1) (10 points) Let n be an arbitrary positive integer. Using the definitions, prove that the function $f : [-1, 1] \to \mathbb{R}$, $f(x) = x^n$ is uniformly continuous.

(2) (10 points) Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that f has a fixed point, i.e., there exists $x \in [0, 1]$, such that $f(x) = x$.

(3) (15 points) Let $f, g : X \to Y$ be continuous functions, and let E be a dense subset of X.
   (a) Prove that $f(E)$ is dense in $f(X)$.
   (b) If $f(p) = g(p)$ for all $p \in E$, prove that $f(x) = g(x)$ for all $x \in X$.

   (In other words, this exercise shows that a continuous function is determined by its values on a dense subset of its domain.)

(4) (15 points) Recall that every rational number $x \in \mathbb{Q}$, $x \neq 0$, can be written uniquely in a reduced form $x = \frac{m}{n}$, where $m, n \in \mathbb{Z}$, $n > 0$. For $x = 0$, take $n = 1$. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q}, \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q}. \end{cases}$$

   (a) Prove that f is continuous at every irrational point.
   (b) Prove that f is discontinuous at every rational point.

(5) (10 points) Let X be a compact subset of $\mathbb{R}$ and $f : X \to \mathbb{R}$ be a function. Define the *graph* of f to be the set

$$\mathcal{G}(f) = \{(x, f(x)) : x \in X\}.$$

   Prove that f is continuous on X if and only if $\mathcal{G}(f)$ is compact.

   (The same statement holds if $f : X \to Y$, with X, Y arbitrary metric spaces and X compact; you don't need to prove the general statement in this assignment.)

Extra problem: Continuous extensions: ex 13 pp 99-100.