HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 5: DUE THURSDAY, MARCH 8, AT 11:00 IN 2-108.

Please remember to tell us which lecture section you are in (Ciubotaru, Melrose, or Parker).

(1) (20 points) Let ℓ^∞ be the set of bounded sequences of real numbers, i.e., $\underline{a} = \{a_i\}$ such that $\sup\{|a_i| : i = 1, 2, 3, \dots\} < \infty$. Define $d(\underline{a}, \underline{b}) = \sup\{|a_i - b_i| : i = 1, 2, 3, \dots\}$.
   (a) Check that ℓ^∞ is a metric space.
   (b) Show that the unit ball, $\overline{B}(\underline{0}, 1) = \{\underline{a} : d(\underline{0}, \underline{a}) \leq 1\}$, is both closed and bounded.
   (c) Prove that the unit ball is not compact. (Therefore, the Heine-Borel theorem is false in ℓ^∞.) *Hint*: Produce an infinite set in $\overline{B}(\underline{0}, 1)$ with no limit point.

(2) (15 points) Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7.
   (a) Is E dense in $[0, 1]$?
   (b) Is E compact?
   Prove your answers.

(3) (10 points) Let A and B be two connected subsets of a metric space X. Assume that $A \cap B \neq \emptyset$. Prove that $A \cup B$ is also connected.

(4) (10 points) Suppose $\{x_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{x_{n_i}\}$ converges to a point $x \in X$. Prove that the full sequence $\{x_n\}$ converges to x.

(5) (10 points) If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}}, \ n \geq 1,$$

   prove that
   (a) $0 < s_n < 2$ for all $n \geq 1$;
   (b) $\{s_n\}$ converges.

Extra problem: This is for your amusement, and not to be handed in.
Baire's theorem. Let X be a nonempty complete metric space, and $\{G_n\}$ be a sequence of dense open subsets of X. Prove that $\cap_{n=1}^{\infty} G_n$ is not empty. (In fact, the intersection is also dense.)

(Hint: prove first that if $F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$ is a sequence of nonempty closed and bounded sets in X, then $\cap_{n=1}^{\infty} F_n$ is nonempty. Then find a shrinking sequence of neighborhoods E_n, such that $\overline{E}_n \subset G_n$.)