HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007
ASSIGNMENT 5: Solutions.

1. Let ℓ^∞ be the set of bounded sequences of real numbers, i.e., $\underline{a} = \{a_i\}$ such that $\sup\{|a_i| : i = 1, 2, 3, \dots\} < \infty$. Define $d(\underline{a}, \underline{b}) = \sup\{|a_i - b_i| : i = 1, 2, 3, \dots\}$.

(1) Check that ℓ^∞ is a metric space.
(2) Show that the unit ball, $\overline{B}(\underline{0}, 1) = \{\underline{a} : d(\underline{0}, \underline{a}) \leq 1\}$, is both closed and bounded.
(3) Prove that the unit ball is not compact. (Therefore, the Heine-Borel theorem is false in ℓ^∞.) *Hint*: Produce an infinite set in $\overline{B}(\underline{0}, 1)$ with no limit point.

Solution. (1) We need to check that d satisfies the axioms of a metric function:

(i) $d(\underline{a}, \underline{b}) \geq 0$ being defined as the supremum over a set of nonnegative numbers. Clearly $d(\underline{a}, \underline{a}) = 0$. Moreover, if $\underline{a} \neq \underline{b}$, there exists i such that $a_i \neq b_i$. Then $d(\underline{a}, \underline{b}) \geq |a_i - b_i| > 0$.

(ii) $d(\underline{a}, \underline{b}) = d(\underline{b}, \underline{a})$ since $|a_i - b_i| = |b_i - a_i|$ for every i.

(iii) We check the triangle inequality for the sequences $\underline{a}, \underline{b}, \underline{c}$. For every i,

$$|a_i - c_i| \leq |a_i - b_i| + |b_i - c_i| \leq d(\underline{a}, \underline{b}) + d(\underline{b}, \underline{c}).$$

(The first inequality is the triangle inequality in $\mathbb{R}^1$, and the second is the fact that $x \leq supE$ for every element x is a bounded set E.) This shows that $d(\underline{a}, \underline{b}) + d(\underline{b}, \underline{c})$ is an upper bound for $\{|a_i - c_i| : i \geq 1\}$, and by therefore by the property of the supremum

$$d(\underline{a}, \underline{b}) + d(\underline{b}, \underline{c}) \geq d(\underline{a}, \underline{c}).$$

(ii) This comes from a general fact, true in every metric space: any closed ball is closed and bounded. See your class notes or Rudin.

(iii) Consider the sequence $\{\underline{x}_n\}$, defined by

$$\underline{x}_n = (\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots, 0, \dots).$$

For every n, $d(\underline{0}, \underline{x}_n) \leq 1$, so $\underline{x}_n \in \overline{B}(\underline{0}, 1)$.

The key observation is that whenever $n \neq m$, $d(\underline{x}_n, \underline{x}_m) = 1$. It implies that no subsequence of $\{\underline{x}_n\}$ is convergent (maybe the easiest way to see this is because no subsequence is Cauchy), and therefore the set $\{\underline{x}_n\}$ has no limit point, and thus $\underline{B}(\underline{0}, 1)$ cannot be compact

2. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7.

(1) Is E dense in $[0, 1]$?
(2) Is E compact?

Prove your answers.

Solution. (1) No. Since every element $x \in E$ contains only 4 and 7 in the decimal expansion, $\inf E = 0.444\ldots$ and $\sup E = 0.777\ldots$. Therefore E is not dense in $[0,1]$. For example, the open interval $(0, 0.444\ldots)$ does not contain any point of E.

(2) Yes. Note that $[0,1]$ is compact, and therefore it is sufficient to check only if E is closed in $[0,1]$. By way of contradiction, let's assume that $x^* \in [0,1]$ is a limit point of E, but $x^* \notin E$. Therefore, there exists a first decimal digit of x different than 4 or 7. Assume this decimal is a at position n. Choose $\epsilon = 10^{-n-1}$. If $a \in \{0,1,2,9\}$, then the distance between x^* and any point of E is at least 10ϵ. Otherwise, the distance between x^* and any point of E is at least 2ϵ. In any case, $(x^* - \epsilon, x^* + \epsilon) \cap E = \emptyset$, which is a contradiction.

3. Let A and B be two connected subsets of a metric space X. Assume that $A \cap B \neq \emptyset$. Prove that $A \cup B$ is also connected.

Solution. Let us assume, by contradiction, that $A \cup B$ is disconnected. Then there exist two nonempty separated subsets C and D, such that $A \cup B = C \cup D$. (Recall that "separated" means that $\overline{C} \cap D = \emptyset$ and $C \cap \overline{D} = \emptyset$.)

Define $A_1 = A \cap C$ and $A_2 = A \cap D$, and similarly B_1, B_2. Clearly $A = A_1 \cup A_2$. Moreover, A_1 and A_2 are separated, e.g., $\overline{A_1} \cap A_2 \subset \overline{C} \cap D = \emptyset$. Since A is connected, necessarily either A_1 or A_2 must be empty, and the other one all of A. Assume, without loss of generality, that $A_1 = \emptyset$, and $A_2 = A$. Equivalently, $A \cap C = \emptyset$ (which immediately implies $C \subset B$) and $A \subset D$.

The same argument with B_1 and B_2 suggests that one of B_1 or B_2 must be empty. Since $B \cap C = C \neq \emptyset$, the only choice is $B_2 = B \cap D = \emptyset$. Now this implies that $D \subset A$ and $B \subset C$.

But then $A = D$ and $B = C$, which is a contradiction since C and D are separated, but $A \cap B \neq \emptyset$. (Note that the hypothesis in the problem may be weakened by requiring only that A and B not be separated.)

4. Suppose $\{x_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{x_{n_i}\}$ converges to a point $x \in X$. Prove that the full sequence $\{x_n\}$ converges to x.

Solution. Let $\epsilon > 0$ be arbitrary. Since $\{x_{n_i}\}$ converges, there exists $N_1 > 0$ such that $d(x_{n_i}, x) < \frac{\epsilon}{2}$, for all $n_i > N_1$. The fact that $\{x_n\}$ is a Cauchy sequence implies that there exists $N > 0$ such that $d(x_n, x_m) < \frac{\epsilon}{2}$, for all $n \geq N$ and $m \geq N$.

Let n_{i0} be such that $n_{i0} \geq \max\{N_1, N\}$. This guarantees that $d(x_{n_{i0}}, x) < \frac{\epsilon}{2}$ and $d(x_n, x_{n_{i0}}) < \frac{\epsilon}{2}$, for all $n \geq N$. Then, by the triangle inequality, for every $n \geq N$,

$$d(x_n, x) \leq d(x_n, x_{n_{i0}}) + d(x_{n_{i0}}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, x_n converges to x.

5. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}},\ n \geq 1,$$

prove that

(1) $0 < s_n < 2$ for all $n \geq 1$;
(2) $\{s_n\}$ converges.

Solution. We will show that $\{s_n\}$ is both monotonically increasing and bounded. Therefore it is convergent.

(a) We'll prove by induction on n that $0 < s_n < 2$ for all n. This is satisfied for $n = 1$ since $s_1 = \sqrt{2}$. Let us assume that $0 < s_n < 2$. Then $s_{n+1} = \sqrt{2 + \sqrt{s_n}} < \sqrt{2 + \sqrt{2}} < \sqrt{2 + 2} = 2$. Clearly $s_{n+1} > 0$ also.

(b) Now let's show that $\{s_n\}$ is increasing by induction. First, $s_1 = \sqrt{2}$ and $s_2 = \sqrt{2 + \sqrt{\sqrt{2}}}$, and so $s_2 > s_1$. Assume $s_n > s_{n-1}$. Then $s_{n+1} = \sqrt{2 + \sqrt{s_n}} > \sqrt{2 + \sqrt{s_{n-1}}} = s_n$.

Extra problem: This is for your amusement, and not to be handed in.

Baire's theorem. Let X be a nonempty complete metric space, and $\{G_n\}$ be a sequence of dense open subsets of X. Prove that $\cap_{n=1}^{\infty} G_n$ is not empty. (In fact, the intersection is also dense.)

Solution. We'll prove first a lemma.

Lemma. *Let X be a complete metric space, and $F_1 \supset F_2 \subset F_3 \supset \cdots \supset F_n \supset F_{n+1} \supset \dots$ a chain of nonempty closed sets such that $\lim_{n\to\infty} diam\ F_n = 0$. Then $\cap_{n=1}^{\infty} F_n \neq \emptyset$.*

Proof. For every $n \geq 1$, choose $x_n \in F_n$ arbitrarily.Note that, if $n < m$, then $d(x_n, x_m) \leq diam\ F_n$, since $x_n, x_m \in F_n$. Let $\epsilon > 0$ be given. There exists $N > 0$ such that $diam\ F_N < \epsilon$. For all $n \geq N$ and $m \geq N$, we have $x_n \in F_N$ and $x_m \in F_N$, and therefore $d(x_n, x_m) < \epsilon$. This shows that the sequence $\{x_n\}$ is Cauchy in X, and, X being complete, it converges to some $x \in X$.

We claim that $x \in F_m$ for every m. Indeed, for every m, the point x is also the limit of the sequence $\{x_m, x_{m+1}, x_{m+2}, \dots\}$. This sequence is contained in F_m, which is closed, and thus $x \in F_m$.

In conclusion, $x \in \cap_{n=1}^{\infty} F_n$. (In fact, $\cap_{n=1}^{\infty} F_n = \{x\}$, otherwise there will be a contradiction with $\lim_{n\to\infty} diam\ F_n = 0$.

$\square$

Now, let $\{G_n\}$ be a sequence of dense open subsets in X. Recall that a set G is *dense* in X if $G \cap E \neq \emptyset$ for all open nonempty subsets $E \subset X$.

To show that $\cap_{n=1}^{\infty} G_n \neq \emptyset$, we will construct a sequence of neighborhoods V_n, $n \geq 1$, such that $\overline{V}_n \subset G_n$, and $V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} \supset \dots$, and $\lim_{n\to\infty} diam\ V_n = 0$. Then by the previous lemma, applied to $\{\overline{V}_n\}$, we would find that $\cap_{n=1}^{\infty} G_n \supseteq \cap_{n=1}^{\infty} \overline{V}_n \neq \emptyset$.

The neighborhoods V_n are constructed as follows. First choose an arbitrary point $x \in G_1$. Since G_1 is open, there exists a ball $B(x, \delta) \subset G_1$. Set $V_1 = B(x, \frac{\delta}{2})$, so that $\overline{V}_1 \subset B(x, \delta) \subset G_1$. Note that *diam* $V_1 = \frac{\delta}{2}$.

Since G_2 is dense, there exists a point y in the open set $G_2 \cap V_1$. We find a ball $B(y, r) \subset G_2 \cap V_1$. Let V_2 be the open ball $B(y, r')$, where $r' = \min\{\frac{r}{2}, \frac{\delta}{2^2}\}$. This choice ensures that $\overline{V}_2 \subset B(y, r) \subset G_2$, and $diam V_2 \leq \frac{\delta}{2^2}$.

We continue in this fashion, constructing inductively V_n using G_n and V_{n-1}. Then $\overline{V}_n \subset G_n$, and *diam* $V_n \leq \frac{\delta}{2^n}$. The sequence $\{V_n\}$ has the desired properties. This concludes the proof of $\cap_{n=1}^{\infty} G_n \neq \emptyset$.

To show that, in fact, $\cap_{n=1}^{\infty} G_n$ is actually dense is not much harder. One only needs to alter the above proof in the initial step. We need to prove that for every open nonempty set $E \subset X$, we have $E \cap \cap_{n=1}^{\infty} G_N \neq \emptyset$. The only change is that we choose $x \in G_1 \cap E$ (this is possible, since G_1 is dense) and the ball $B(x, \delta) \subset G_1 \cap E$ (this is possible since $G_1 \cap E$ is open). The rest is the same.