HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 4: DUE THURSDAY 1 MARCH, AT 11:00 IN 2-108.

Please remember to tell us which lecture section you are in (Ciubotaru, Melrose, or Parker).

(1) (20 points) For E any subset of a metric space X, define E° to be the union of all open sets contained inside E.

(a) Show that E° is open.

E° is a union of open sets, and is therefore open, as an arbitrary union of open sets is open.

(b) Show that E° is equal to the set of all interior points of E (in other words the set of points $p \in E$ so that there exists some $r > 0$ so that the open ball $B_r(p)$ of radius r centered on p is contained in E .)

If $p \in E^\circ$, then there exists an open ball $B_r(p)$ centered on p contained inside E°. As E° is a union of sets contained inside E, $E^\circ \subset E$. Therefore, $B_r(p) \subset E$, and p is an interior point of E. We have now shown that E° is contained in the set of interior points of E, and must show that E° contains all interior points.
Suppose that p is an interior point of E. Therefore there is some open ball $B_r(p)$ containing p which is contained in E. This open ball is an open subset of E, so E° is the union of $B_r(p)$ with all the other open subsets of E. This means that $E^\circ \supset B_r(p)$, and therefore $p \in E$.

(c) Show that the complement of E°, $X - E^\circ$ is the closure of the complement of E.

E° is union of all open sets contained inside E. The complement of E° is therefore the intersection of the compliment of all open sets contained inside E. U is an open set contained inside E if and only if $X - U$ is a closed set containing $X - E$. The complement of E° is therefore the intersection of all closed sets containing the complement of E, which is the closure of the complement of E.

(d) Do E° and E have the same closures?

No. If $E := \{0\} \subset \mathbb{R}$, then $E^\circ = \emptyset$. The closure of this is $\emptyset$. On the other hand, the closure of E is $\{0\}$.

(2) (10 points) Show that the subset of $\mathbb{R}$ given by

$$E := \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$$

is compact.

Let $\{U_\alpha \text{ for } \alpha \in A\}$ be some open cover of $E \subset \mathbb{R}$. We must show that there exists a finite subcover.

First, at least one of these open sets U_{α_0} must contain 0. As U_{α_0} is open and contains 0, there exists some $r > 0$ so that $(-r, r) \subset U_\alpha$.

We know that there exists some number $n \in \mathbb{N}$ so that $n > \frac{1}{r}$. This means that $\frac{1}{m} \subset U_{\alpha_0}$ for all $m \geq n$.

For any $k \in \mathbb{N}$, there must be some $U_{\alpha_k} \in \{U_\alpha\}$ so that $\frac{1}{k} \subset U_{\alpha_k}$. The finite cover of E which we want is given by

$$\{U_{\alpha_0}, U_{\alpha_1}, \dots, U_{\alpha_{n-1}}\}$$

Another way to prove that E is compact is by proving that it is a closed, bounded subset of $\mathbb{R}$. For a full score, the proof that E is closed must be included.

(3) (10 points) Show that if $C_\alpha \subset X$ is a compact subset of X for all $\alpha \in A$, then

$$C := \bigcap_{\alpha \in A} C_\alpha$$

is compact.

Recall that any compact subset is closed. C is an intersection of closed sets, and is therefore closed. For any C_α, C is a closed subset of C_α. As closed subsets of a compact space are compact, C is compact.

(4) (20 points) Suppose that the metric space X has a countable dense subset Q (In other words, a countable subset Q so that the closure $\bar{Q} = X$.)

(a) Show that any open subset of X is a union of open balls $B_r(q)$ where $r \in \mathbb{Q}$ and $q \in Q$.

The fact that $Q \subset X$ is dense implies that any nonempty open set $U \subset X$ must contain some member of Q. (This follows from the fact that the complement of U is a closed set which is a proper subset of the closure of Q, and thus must not contain all of Q.)
Suppose that $p \in U$ where U is an open set. We must prove that $p \in B_r(q)$ for some $r \in \mathbb{Q}$ and $q \in Q$ so that $B_r(q) \subset U$. Then U is equal to the union of all these open balls. First note that there exists some $\epsilon > 0$ so that $B_\epsilon(p) \subset U$. As $B_{\frac{\epsilon}{2}}$ is an open set, it must contain some $q \in Q$. Choose r to be a rational number so that $d(p, q) < r < \frac{\epsilon}{2}$. Then $p \in B_r(q)$ and $B_r(q) \subset B_\epsilon(p) \subset U$. This follows from the fact that for all $x \in B_r(q)$, the triangle inequality tells us that

$$d(p, x) \leq d(p, q) + d(q, x) < \epsilon$$

(b) Show that any open cover of X has a sub cover which is either countable or finite.
First note that the set $\mathcal{B}$ of open balls $B_r(q)$ where $r \in \mathbb{Q}$ and $q \in Q$ is a countable union of countable sets, and is hence countable.
Suppose that $\{U_\alpha\}$ is an open cover of X. We have shown that each of these open sets is some union of open balls $B_r(q) \in \mathcal{B}$. Define $E \subset \mathcal{B}$ to be the set of such open balls that is contained in one of these $\{U_\alpha\}$. As it is a subset of a countable set, it is either contable or finite. The union of all $B_r(q) \in E$ is X. For each $B_r(q) \in E$, choose some U_α from our cover contining $B_r(q)$. The set of all such U_α is equal to the union of all $B_r(q) \in E$, which is X. The set of these chosen U_α is a subcover with cardinality less than or equal to E, and is therefore either countable or finite.

Extra problem: This is for your amusement, and not to be handed in.

(1) Call a metric d on $\mathbb{R}^n$ compatible with the vector space structure if

$$d(x+z, y+z) = d(x,y) \text{ for all } x, y, z \in \mathbb{R}^n$$

and

$$d(\lambda x, \lambda y) = \lambda d(x,y) \text{ for } \lambda \in [0, \infty)$$

Prove that any such metric puts the same topology on $\mathbb{R}^n$ as the Euclidean metric, in the sense that a subset of $\mathbb{R}^n$ is open with this metric if and only if it is open with the Euclidean metric.

We shall show that there exist constants $c_1 > 0$ and $c_2 > 0$ so that

$$c_1 d(x,y) \leq \|x-y\| \leq c_2 d(x,y)$$

This will imply that inside every open Euclidean ball there is an open d ball, which means that any Euclidean open set is an open set with the metric d. The other inequality implies that inside every open d ball there is an open Euclidean ball, which implies that any set open with the metric d is open with the Euclidean metric.

First let $\lambda = \min_i d(0, e_i) > 0$ where e_i is the ith standard basis vector of $\mathbb{R}^n$ (with the ith coordinate 1 and the other coordinates zero.) Then by the triangle inequality, we have that

$$d(0,x) \leq \sum_{i=1}^{n} d(0, x_i e_i)$$

where x_i is the ith coordinate of x. If $x_i > 0$, then $d(0, x_i e_i) = x_i d(0, e_i)$. If $x_i < 0$, then $d(0, x_i e_i) = d(-x_i e_i, 0) = -x_i d(0, e_i)$. Therefore,

$$d(0,x) \leq \sum_{i=1}^{n} |x_i| \lambda \leq n\lambda \|x\|$$

So if we set $c_1 = \frac{1}{n\lambda}$, the we have our first inequality:

$$c_1 d(x,y) = c_1 d(0, x-y) \leq \|x-y\|$$

What we now want is some c_2 so that $c_2 d(x,y) \geq \|x-y|$, or equivalently, $d(x,0) \geq \frac{1}{c_2}$ whenever $\|x\| = 1$. Consider the unit sphere in $\mathbb{R}^n$,

$$S := \{x \in \mathbb{R}^n \text{ so that } \|x\| = 1\}$$

This is a closed, and bounded subset of $\mathbb{R}^n$, so we know that it is compact. Lets try to use this compactness to construct c_2. For each $x \in S$, consider the Euclidean open set

$$U_x := \left\{ y \in \mathbb{R}^n \text{ so that } \|x - y\| < \frac{c_1 d(x,0)}{2} \right\}$$

Note that for $y \in U_x$, we have that

$$d(y,0) \geq d(x,0) - d(x,y) > \frac{d(x,0)}{2}$$

Note also that $\{U_x, x \in S\}$ is an Euclidean open cover for S. Therefore, there exists a finite subcover $\{U_{x_1}, \ldots, U_{x_k}\}$. Now we can define c_2 by

$$\frac{1}{c_2} := \frac{1}{2} \min\{d(x_1,0), \ldots, d(x_k,0)\}$$

We now have the inequality

$$c_2 d(x,0) > 1 \text{ for all } x \in S$$

and our second inequality follows from the fact that our metric is compatible with the vector space structure.

$$c_2 d(x,y) \geq \|x - y\|$$

Note that we used that n is finite. This is not true in an infinite dimensional vector space.