HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 4: DUE THURSDAY 1 MARCH, AT 11:00 IN 2-108.

Please remember to tell us which lecture section you are in (Ciubotaru, Melrose, or Parker).

(1) (20 points) For E any subset of a metric space X, define E° to be the union of all open sets contained inside E.
   (a) Show that E° is open.
   (b) Show that E° is equal to the set of all interior points of E (in other words the set of points $p \in E$ so that there exists some $r > 0$ so that the open ball $B_r(p)$ of radius r centered on p is contained in E .)
   (c) Show that the complement of E°, $X - E^\circ$ is the closure of the complement of E.
   (d) Do E° and E have the same closures?

(2) (10 points) Show that the subset of $\mathbb{R}$ given by

$$E := \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$$

is compact.

(3) (10 points) Show that if $C_\alpha \subset X$ is a compact susbset of X for all $\alpha \in A$, then

$$C := \bigcap_{\alpha \in A} C_\alpha$$

is compact.

(4) (20 points) Suppose that the metric space X has a countable dense subset Q (In other words, a countable subset Q so that the closure $\bar{Q} = X$.)
   (a) Show that any open subset of X is a union of open balls $B_r(q)$ where $r \in \mathbb{Q}$ and $q \in Q$.
   (b) Show that any open cover of X has a sub cover which is either countable or finite.

Extra problem: This is for your amusement, and not to be handed in.

(1) Call a metric d on $\mathbb{R}^n$ compatible with the vector space structure if

$$d(x + z, y + z) = d(x, y) \text{ for all } x, y, z \in \mathbb{R}^n$$

and

$$d(\lambda x, \lambda y) = \lambda d(x, y) \text{ for } \lambda \in [0, \infty)$$

Prove that any such metric puts the same topology on $\mathbb{R}^n$ as the Euclidean metric, in the sense that a subset of $\mathbb{R}^n$ is open with this metric if and only if it is open with the Euclidean metric.