HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007
ASSIGNMENT 3: DUE THURSDAY 22 FEB, AT 11:00 IN 2-108.

Each problem is worth 10 points.

(1) Let $E \subset \mathbb{R}$ be an uncountable subset. Show that $[-n, n] \cap E$ must be uncountable for some $n \in \mathbb{N}$.

(2) Construct a bounded subset of $\mathbb{R}$ with exactly 3 limit points.

(3) Show that if d is a distance function on a set M then so is

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

(4) Is every point of an open subset $E \subset \mathbb{R}^2$ a limit point of E?

(5) What are the real limit points of $E = \{q \in \mathbb{Q}; 0 < q < 1\}$?

(6) Give an example of a countable collection $U_j \subset \mathbb{R}$, $j \in \mathbb{N}$, of open sets with $\bigcap_{j \in \mathbb{N}} U_j$ an uncountable *closed* set.

(7) Let $M = [0, 1]$ be a metric space with the discrete metric, $d(x, y) = 1$ if $x \neq y$, $d(x, x) = 0$. What are the open sets in M?

* Extra Problem – for your amusement only.

If you think of the existence of a 1-1 map from A into B as implying that A is 'not bigger than' B, then this exercise shows that if A is not bigger than B and B is not bigger than A, then A and B have the same cardinality.

Schroeder-Bernstein Theorem If A and B are any two sets, f is a 1-1 map from A into B and g is a 1-1 map from B into A, then there exists a map F from A to B which is 1-1 and onto, i.e., $A \sim B$.

Here is an outline of the proof due to Birkhoff and MacLane which you are invited to complete.

(a) Define 'ancestors' as follows: Consider $a \in A$, if $a \in g(B)$ then we call $g^{-1}(a)$ the first ancestor of a (we call a itself the zeroth ancestor of a). If $g^{-1}(a) \in f(A)$ then we call $f^{-1}(g^{-1}(a))$ the second ancestor of a. If this is in the image of g, then we call $g^{-1}(f^{-1}(g^{-1}(a)))$ the third ancestor of a and so on. Show that this divides A into three disjoint subsets: A_∞ consisting of the elements that have infinitely many ancestors, A_e consisting of the elements that have an even number of ancestors (including none), and A_o consisting of the elements that have an odd number of ancestors.

(b) Show that you can partition B into three similar subsets: B_∞, B_e and B_o.

(c) Identify $f(A_\infty)$, $f(A_e)$.

(d) Define

$$F(a) = \begin{cases} f(a) & \text{if } a \in A_\infty \cup A_e \\ g^{-1}(a) & \text{if } a \in A_o \end{cases}$$

and show that F is a 1-1 correspondence between A and B.