HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 SOLUTION TO ASSIGNMENT 3: WAS DUE THURSDAY 22 FEB, AT 11:00 IN 2-108.

Each problem is worth 10 points.

(1) Let $E \subset \mathbb{R}$ be an uncountable subset. Show that $[-n, n] \cap E$ must be uncountable for some $n \in \mathbb{N}$.

Solution: By the Archimedian property of the real numbers,

$$E = \bigcup_{n \in \mathbb{N}} ([-n, n] \cap E)$$

for any $E \subset \mathbb{R}$. This is a countable union, so if each $[-n, n] \cap E$ is countable so is E. Thus if E is uncountable, at least one of the $[-n, n] \cap E$ must be uncountable.

(2) Construct a bounded subset of $\mathbb{R}$ with exactly 3 limit points.

Solution: Consider the set

$$E = \{\frac{1}{n}, n \in \mathbb{N}\} \cup \{1 + \frac{1}{n}, n \in \mathbb{N}\} \cup \{2 + \frac{1}{n}, n \in \mathbb{N}\} \subset \mathbb{R}.$$

The points $e = 0$, 1 and 2 are limit points since the ball $B(e, r)$ with radius $r > 0$ around each point contains the infinite set $\{e + \frac{1}{n}; rn > 1\}$ in each case. No other point $x \in \mathbb{R}$ is a limit point since there exists $r > 0$ such that $|x - e| \geq 2r$ for $e = 0$, 1 and 2 and then $B(x, r) \cap E$ is contained in the finite subset consisting of the points $e + \frac{1}{n}$ where $nr < 1$. Thus E has exactly three limit points.

(3) Show that if d is a distance function on a set M then so is

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Solution: Certainly $d' : M \times M \longrightarrow \mathbb{R}$ satisfies the first two conditions require of a distance. Namely $d'(x, y) \geq 0$ for all x, $y \in M$, $d'(x, y) = 0$ implies $d(x, y) = 0$ implies $x = y$ and $d'(x, y) = d'(y, x)$ for all x, $y \in M$. Thus it remains to check the triangle inequality. We do this by multiplying through by the all three denominators, so we wish to show that

$$(1+d(x,z))(1+d(z,y))d(x,y) \leq (1+d(x,y))(1+d(z,y))d(x,z)+(1+d(x,y))(1+d(x,z))d(z,y)$$

for any three points. From the triangle inequality for d, or trivially, the three inequalities

$$d(x, y) \leq d(x, z) + d(z, y)$$

$$d(x,z)d(x,y) + d(z,y)d(x,y) \leq d(x,y)d(x,z) + d(z,y)d(x,z) + d(x,y)d(z,y) + d(x,z))d(z,y)$$

$$d(x,z)d(z,y)d(x,y) \leq d(x,y)d(z,y)d(x,z) + d(x,y)d(x,z)d(z,y)$$

always hold. Adding them gives the triangle inequality for d'.

Comment:- Notice that each open ball for the distance d is an open ball for the distance d' (with a different radius) and conversely and that open sets and closed sets are the same in the two metric spaces (M, d) and (M, d')

– so they have the same 'topology'. However, d' is bounded above by 1 so 'boundedness is not a topological property'.

(4) Is every point of an open subset $E \subset \mathbb{R}^2$ a limit point of E?

Solution:- Yes. If E is empty then there are no limit points. If E is non-empty and $x \in E$ then for $r > 0$ sufficiently small $B(x, r) \subset E$ is infinite, and hence is infinite for all $r > 0$, so x is a limit point.

(5) What are the real limit points of $E = \{q \in \mathbb{Q}; 0 < q < 1\}$?

Solution:- The set of real limit points is the closed interval $[0, 1] \subset \mathbb{R}$. Namely if $x \in (0, 1]$ then for $r > 0$ by the Archimedian property of the reals there exists $q \in \mathbb{Q}$ with $\min(x-r, 0) < q < x$ (since $\min(x-r, 0) < x$). Thus $x \in E'$. Similarly for 0, $B(0, r) \cap E \supset \{\frac{1}{n}\}$ if $nr > 1$ so 0 is also a limit point. No point $x \in \mathbb{R} \setminus [0, 1]$ can be a limit point since if $r = \min(|x|, |x - 1|) > 0$ then $B(x, r) \cap E = \emptyset$.

(6) Give an example of a countable collection $U_j \subset \mathbb{R}$, $j \in \mathbb{N}$, of open sets with $\bigcap_{j\in\mathbb{N}} U_j$ an uncountable *closed* set.

Solution:- An easy one is to take $U_j = \mathbb{R}$ for all j. We had in mind something more like the set $U_j = (-\frac{1}{j}, 1 + \frac{1}{j})$ for which the intersection is $[0, 1]$.

(7) Let $M = [0, 1]$ be a metric space with the discrete metric, $d(x, y) = 1$ if $x \neq y$, $d(x, x) = 0$. What are the open sets in M?

Solution: Any subset is open. Namely, if $E \subset M$ and $x \in E$ then $B(x, \frac{1}{2}) = \{x\}$ since all other points are distant 1 from x. Thus $B(x, \frac{1}{2}) \subset E$ which is therefore open.

* Extra Problem – for your amusement only.

If you think of the existence of a 1-1 map from A into B as implying that A is 'not bigger than' B, then this exercise shows that if A is not bigger than B and B is not bigger than A, then A and B have the same cardinality.

Schroeder-Bernstein Theorem If A and B are any two sets, f is a 1-1 map from A into B and g is a 1-1 map from B into A, then there exists a map F from A to B which is 1-1 and onto, i.e., $A \sim B$.

Here is an outline of the proof due to Birkhoff and MacLane which you are invited to complete.

(a) Define 'ancestors' as follows: Consider $a \in A$, if $a \in g(B)$ then we call $g^{-1}(a)$ the first ancestor of a (we call a itself the zeroth ancestor of a). If $g^{-1}(a) \in f(A)$ then we call $f^{-1}(g^{-1}(a))$ the second ancestor of a. If this is in the image of g, then we call $g^{-1}(f^{-1}(g^{-1}(a)))$ the third ancestor of a and so on. Show that this divides A into three disjoint subsets: A_∞ consisting of the elements that have infinitely many ancestors, A_e consisting of the elements that have an even number of ancestors (including none), and A_o consisting of the elements that have an odd number of ancestors.

(b) Show that you can partition B into three similar subsets: B_∞, B_e and B_o.

(c) Identify $f(A_\infty)$, $f(A_e)$.

(d) Define

$$F(a) = \begin{cases} f(a) & \text{if } a \in A_\infty \cup A_e \\ g^{-1}(a) & \text{if } a \in A_o \end{cases}$$

and show that F is a 1-1 correspondence between A and B.