HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 2 SOLUTIONS.

(1) (10 pts total) Rudin, Chapter 2, problems 2,3,4. Solutions to these problems can be found at http://ocw.mit.edu/OcwWeb/Mathematics/18-100BAnalysis-IFall2002/Assignments/ so the main point here is to be clear, concise and correct!

(2) (10 pts) Show the supremum of the set $A = \{\frac{x}{\sqrt{1+x^2}}; x \in \mathbb{R}\}$ exists and find it (you need to prove that your answer is correct).

Solution. For $x \in \mathbb{R}$, $x^2 < 1 + x^2$ and hence $x < \sqrt{1+x^2}$ (since $x \geq \sqrt{1+x^2}$ would imply $x^2 \geq 1 + x^2$). This shows that

$$\frac{x}{\sqrt{1+x^2}} < 1$$

which means that 1 is an upper bound for A. Thus A is bounded above and we will show that $1 = \sup A$. To do so it suffices to show that there is no smaller upper bound, that $t < 1$ implies that t is not an upper bound. This is clear if $t \leq 0$ since $0 \in A$. If $0 < t < 1$ then choose s with $0 < t < s < 1$ (using the Archmidean property) and consider

$$x = \frac{s}{\sqrt{1-s^2}}$$

which exists because $1 - s^2 > 0$. Then

$$\frac{x}{\sqrt{1+x^2}} = s > t$$

so t is not an upper bound.

(3) (10 pts) Suppose A and B are bounded, non-empty, subsets of $\mathbb{R}$ and that $A \cap B \neq \emptyset$. Show that

$$\sup(A \cap B) \leq \min(\sup A, \sup B).$$

Solution: An upper bound for A is an upper bound for $A \cap B$ which is therefore bounded above, and non-empty by assumption, so $c = \sup(A \cap B) \in \mathbb{R}$ exists. Since $\sup A$ is an upper bound for A, it is an upper bound for $A \cap B$ so $c \leq \sup(A)$. The same is true for B so $c \leq \min(\sup A, \sup B)$ as desired.

(4) (10 pts) Show that the set of all rational sequences $a : \mathbb{N} \longrightarrow \mathbb{Q}$ is uncountable. Prove that the subset of terminating sequences, i.e. such that $a_n = 0$ for $n > N$ where N may vary with the sequence, is countable.

Solution: The set of rational sequences contains the set of sequences with values in $0, 1$ which is shown to be uncountable in Rudin, so it must also be uncountable. Consider the set, $\mathcal{S}_N$ of those rational sequences which terminate after N terms, i.e. $a_n = 0$ for $n > N$. This is equivalent to $\mathbb{Q}^N$ where the equivalence is just the map to the first N terms. This is shown to be countable in Rudin (or in class). The set of all terminating rational sequences is

$$\bigcup_{N=1}^{\infty} \mathcal{S}_N$$

so as a countable union of countable sets it is also countable.

Extra problem: This is for your amusement, not to be handed in.

(1) Two fields are *isomorphic* if there is a 1-1 onto map between them which takes sums to sums and products to products. Show that such a map must take 0 to 0 and 1 to 1. Show that the set of real numbers of the form $r+s\sqrt{2}$, where r, $s \in \mathbb{Q}$, is a field. Is it isomorphic to $\mathbb{Q}$?

Solution: You can check that it is a field by going through the axioms! In fact it is a subset of the real numbers so only the existence of sums, products, additive and multiplicative inverses needs to be checked. For products

$$(r+s\sqrt{2})(r'+s'\sqrt{2}) = rr' + 2ss' + (rs' + sr')\sqrt{2}$$

and the inverse of $r + s\sqrt{2}$ is $r' + s'\sqrt{2}$ where $r' = r/(r^2 - 2s^2)$, $s' = -s/(r^2 - 2s^2)$ which exists whenever one of r and s is non-zero (since they are rational so $r^2 = 2s^2$ is impossible unless both r and s vanish.

A field isomorphism would have to carry products to products. If it existed then $\sqrt{2}$ would be mapped to some rational q which would have to satisfy $q^2 = 2$ which is impossible.