SOLUTION TO HOMEWORK 1 FOR 18.100B AND 18.100C, SPRING 2007 WAS DUE THURSDAY 8 FEB, AT 11:00 IN 2-108.

Since you will have had only one lecture by the time this homework is due, it is very short. Although there are three questions they are very closely related! To get full marks you must write out the arguments needed carefully, succinctly and completely.

(1) Show that there is no rational number, q, with the property that $q^2 = 3$.

Solution: Suppose there were such a rational number q with $q^2 = 3$. Then $q = m/n$ where m, n are integers which are unique if we require that $n > 0$ and that m and n have no common integral factor greater than 1. Then $q^2 = 3$ implies $m^2 = 3n^2$. This implies that $m = 3k$ for an integer k. Indeed, otherwise $m = 3k + 1$ or $m = 3k + 2$ for some k and then $m^2 = 9k^2 + 6k + 4 = 3(3k^2 + 2k + 1) + 1$ or $m^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ is not divisible by 3. Thus $n^2 = 3k^2$ and hence n is also divisible by 3, contradicting the condition that m and n have no such common factor.

Comments: It is okay to use the prime decomposition instead. Or you can say that 3 is prime so if 3 divides m^2 it divides one of the factors, i.e. m. Just saying m^2 is divisible by 3 implies m is divisible by 3 is not really good enough.

(2) Using the supremum property of the real numbers show that the set of rational numbers ($\mathbb{Q}$ denotes the set of rational numbers)

$$A = \{q \in \mathbb{Q}; q > 0,\ q^2 < 3\}$$

has a supremum as a set of real numbers.

Solution: Since A is a set of rational numbers, it is a set of real numbers. Now, A is non-empty, since $1^2 = 1 < 3$, so $1 \in A$. Furthermore, A is bounded above by 2 since if $q \geq 2$ then $q^2 \geq 4 > 3$ so $q \notin A$. Thus, by the 'Least upper bound' property of $\mathbb{R}$, $\sup A$ exists.

Comment. It really isn't good enough to say, q^2 bounded above implies q bounded above without an actual argument.

(3) Denoting the real number in (2) by $x = \sup A$, show that $x^2 = 3$.

Solution: From the discussion above, we see that $x = \sup A$ does exists. From the total order property of the reals there are only three possibilities, $x^2 > 3$, $x^2 < 3$ or $x^2 = 3$, so we just have to show that the first two of these are not possible.

So, first assume that $x^2 > 3$. Consider

$$y = x - \frac{x^2 - 3}{x + 3} = \frac{3x + 3}{x + 3} \Longrightarrow y^2 = \frac{9x^2 + 18x + 9}{x^2 + 6x + 9} = 3 + 6\frac{x^2 - 3}{x^2 + 6x + 9} > 3. \tag{1}$$

Thus $y < x$ but $y^2 > 3$ which means that if $q \in A$ then $q^2 < y^2$ which implies $q < y$ (since $q \geq y > 0$ would imply $q^2 \geq y^2$). This contradicts the assumption that x is the least upper bound, since y is a smaller upper bound.

Secondly, suppose $x^2 < 3$. The same choice of y as in (1) now gives $x < y$ and $y^2 < 3$. But then, by the Archimedean principle there is a rational number q with $x < q < y$, so $q^2 < y^2 < 3$ and so $q \in A$, showing that x is not an upper bound for A.

Thus the only remaining possibility is that $x^2 = 3$.

Comments: There are of course other choices of y to show that x is not the least upper bound if $x^2 > 3$. In the second case, you do need to find a rational greater than x but with square less than 3 – there are other ways of doing this too. One thing that we did not accept was the definition of a real as the square of a given number – because the existence of such a number is what we are proving here in the case of 3.