# HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 9: DUE THURSDAY 19 APRIL, AT 11:00 IN 2-108.

(1) (10 points) Show that if $f : [a, b] \longrightarrow \mathbb{R}^k$ is $(n+1)$ times differentiable, and $x, y \in [a, b]$, then

$$f(y) = f(x) + f'(x)(y - x) + \cdots + \frac{f^{(n)}(x)}{n!}(y - x)^n + E(y)$$

where

$$|E(y)| \leq sup_{[a,b]}|f^{(n+1)}|\frac{|y - x|^{n+1}}{(n + 1)!}$$

Hint: try taking the dot product of $E(t)$ with $E(y)$ to get an $\mathbb{R}$ valued function.

Define

$$g(t) := E(t) \cdot E(y)$$

$g : [a, b] \longrightarrow \mathbb{R}$ is $(n + 1)$ times differentiable and the first $n$ derivatives at $x$ are equal to 0. We can therefore apply the one dimensional version of Taylor's theorem to tell us that there exists some point $s$ between $x$ and $y$ so that

$$g(y) = \frac{g^{(n+1)}(s)}{(n + 1)!}(y - x)^{n+1}$$

This tells us that

$$E(y) \cdot E(y) = \frac{E^{(n+1)}(s) \cdot E(y)}{(n + 1)!}(y - x)^{n+1} = \frac{f^{(n+1)}(s) \cdot E(y)}{(n + 1)!}(y - x)^{n+1}$$

Applying the Cauchy Schwartz inequality to this then gives

$$|E(y)|^2 \leq \frac{|f^{(n+1)}(s)|}{(n + 1)!}|E(y)||y - x|^{n+1}$$

so

$$|E(y)| \leq sup_{[a,b]}|f^{(n+1)}|\frac{|y - x|^{n+1}}{(n + 1)!}$$

(2) (10 points) Show that if $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable and there exists some $c$ so that

$$|f'(x)| \leq c|f(x)|$$

then if $f(a) = 0$, $f(x)$ must be 0 for all $x \in [a, b]$.

Hint: don't try to integrate this as $f'$ is not necessarily Riemann integrable...Try to think what the mean value Theorem can tell us about the relationship between the supremum of $f$ and $f'$ on some small neighborhood of $a$.

Consider $f$ on a small interval $[a, a+\epsilon]$ containing $a$. The mean value Theorem tells us that since $f(a) = 0$,

$$\sup_{[a,a+\epsilon]} |f| \leq \epsilon \sup_{[a,a+\epsilon]} |f'|$$

We also have that $|f'| \leq c|f|$. Choosing $\epsilon > 0$ small enough that $\epsilon c < 1$ and combining this with our above inequality, we have

$$\sup_{[a,a+\epsilon]} |f| \leq c\epsilon \sup_{[a,a+\epsilon]} |f|$$

so $f$ is identically 0 on the interval $[a, a+\epsilon]$. We can then repeat this argument for the interval $[a+\epsilon, a+2\epsilon]$ to get that $f$ is identically 0 there. Continuing this process, we get that $f$ is identically 0 on $[a, b]$.

(3) (a) (10 points) Suppose that $f$ and $g$ are real differentiable functions on an interval $[a, b]$ that satisfy the differential equation

$$f'(t) = H(t, f(t))$$

$$g'(t) = H(t, g(t))$$

Where $H : [a, b] \times R \longrightarrow \mathbb{R}$ satisfies the following condition. There exists some constant $c \in \mathbb{R}$ so that

$$|H(t, s_1) - H(t, s_2)| \leq c|s_1 - s_2|$$

Show that if $f(a) = g(a)$ then $f$ and $g$ are equal everywhere.

Hint: apply the previous problem to $f - g$.

Consider the function $h = f - g$. We have that $h$ is differentiable on $[a, b]$, $h(a) = 0$, and

$$|h'(t)| = |H(t, f(t)) - H(t, g(t))| \leq c|f(t) - g(t)| = c|h(t)|$$

Therefore, our previous problem tells us that $h$ is identically 0 on $[a, b]$, which is what we wanted.

(b) (5 points) Show that if $H(t, s) = |s|^{\frac{1}{2}}$ that this is not true.

The functions $f(t) = 0$ and $g(t) = \frac{1}{4}t^2$ are two non equal functions that satisfy this differential equation on the interval $[0, 1]$, and $f(0) = g(0) = 0$.

(4) Denote the space of all continuous real valued functions on $[a, b]$ by $C([a, b])$.

(a) (10 points) Suppose that $\alpha$ is a strictly increasing function on $[a, b]$. (In other words $\alpha(x) > \alpha(y)$ if $x > y$.) Define the following inner product for $f, g \in C([a, b])$:

$$\langle f, g \rangle := \int_a^b fg d\alpha$$

Define the norm of $f \in C([a, b])$ to be

$$\|f\|_2 := (\langle f, f \rangle)^{\frac{1}{2}}$$

Prove the following inequality (called the Cauchy-Schwartz inequality):

$$\langle f, g\rangle \leq \|f\|_2\|g\|_2$$

We shall use the following properties of $\langle f, g\rangle$ which follow directly from the properties of the Riemann-Stiltjes integral:

(i)
$$\langle f, g\rangle = \langle g, f\rangle$$
(ii)
$$\langle cf, g\rangle = \langle f, cg\rangle = c\langle f, g\rangle$$
(iii)
$$\langle f + h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$$
(iv)
$$0 \leq \langle f, f\rangle$$

Now using the above proberties we have

$$0 \leq \langle \lambda_1 f - \lambda_2 g, \lambda_1 f - \lambda_2 g\rangle = \lambda_1^2\langle f, f\rangle + \lambda_2^2\langle g, g\rangle - 2\lambda_1\lambda_2\langle f, g\rangle$$

Setting $\lambda_1 = \langle g, g\rangle$ and $\lambda_2 = \langle f, g\rangle$, we get

$$0 \leq \langle g, g\rangle^2\langle f, f\rangle + \langle f, g\rangle^2\langle g, g\rangle - 2\langle f, g\rangle^2\langle g, g\rangle$$

so if $\langle g, g\rangle \neq 0$

$$\langle f, g\rangle^2 \leq \langle f, f\rangle\langle g, g\rangle$$

This then gives our inequality in the case that $\langle g, g\rangle \neq 0$. It follows similarly if $\langle f, f\rangle \neq 0$, and the above equation with $\lambda_i = \pm 1$ gives the last case that if $\langle f, f\rangle = \langle g, g\rangle = 0$, then $\langle f, g\rangle = 0$.

(b) (10 points) Show that the following defines a metric on $C([a, b])$:

$$d(f, g) := \|f - g\|_2$$

First, it is clear that $d(f, g) = d(g, f)$. We shall now prove that the triangle inequality holds using the Cauchy Schwartz inequality:

$$\|f + g\|_2^2 = \langle f, f\rangle + \langle g, g\rangle + 2\langle f, g\rangle \leq \|f\|_2^2 + \|g\|_2^2 + \|f\|_2\|g\|_2$$

so

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$$

This then implies the triangle inequality:

$$\begin{aligned} d(f, h) &= \|f - h\|_2 \\ &= \|f - g + g - h\|_2 \\ &\leq \|f - g\|_2 + \|g - h\|_2 \\ &= d(f, g) + d(g, h) \end{aligned}$$

It remains to show that if $f \neq g$, then $d(f, g) > 0$. This is the only part for which we need that $f$ and $g$ are continuous and $\alpha$ is strictly increasing.

Suppose that $f \neq g$. Then $f - g$ is continuous because $f$ and $g$ are, and there exists some point $x$ and $c > 0$ so that $f(x) - g(x) \geq c$. Then, there exists some $\delta > 0$ so that for $|y - x| < \delta$, $|f(y) - g(y) - (f(x) - g(x))| < \frac{c}{2}$.

The triangle inequality then tells us that $|f(y) - g(y)| \geq \frac{c}{2}$. There therefore exists some $x_1 < x_2$ so that on the interval $[x_1, x_2]$, $f - g \geq \frac{c}{2}$. Therefore as $(f-g)^2 \geq 0$, for any partition $P$, the upper sum

$$U(P, (f-g)^2, \alpha) \geq \left(\frac{c}{2}\right)^2 (\alpha(x_2) - \alpha(x_1))$$

The right hand side of this is independent of $P$ and strictly positive because $\alpha$ is strictly increasing. Therefore the integral of $(f-g)^2$ is strictly positive, and $d(f,g) > 0$ as required.

(c) (5 points) Give an example showing that this is not a metric on $\mathcal{R}(\alpha)$.

Consider $\alpha = x$ and the function $f$ which is zero everywhere on $[-1, 1]$ apart from 0 where it equals 1. We shall show $d(f, 0) = 0$. For this, we need that the integral of $f^2$ is 0. Note that as $f^2$ is bounded and continuous outside of a finite set, and $x$ is continuous, so $f^2 \in \mathcal{R}$. Note also that for any partition, the lower sum is equal to 0, so the integral of $f^2$ is 0 as required.

(5) (10 points) Recall that any rational in $\mathbb{Q}$ has a unique reduced form $\frac{p}{q}$ where $p$ and $q$ are integers with no common factors and $q > 0$. Define the function $f$ by $f(x) = \frac{1}{q}$ if $x = \frac{p}{q}$ and $f(x) = 0$ for $x \notin \mathbb{Q}$. Prove directly that $f$ is Riemann integrable on $[0, 1]$ and find

$$\int_0^1 f dx$$

Note first that any interval of nonzero size contains irrational numbers, so the lower sums for this integral are always 0. To prove that $f \in \mathcal{R}$, we must construct for any $\epsilon > 0$ a partition so that $U(P, f) < \epsilon$. (Of course, this also shows that our integral is 0).

We do this as follows: for any $\epsilon > 0$ the number of points where $f \geq \frac{\epsilon}{2}$ is some finite number $N$. Choose a partition $P$ of $[0, 1]$ so that each of these points is in the interior of an interval of size at most $\frac{\epsilon}{2N}$. Then, as $f$ is bounded by 1, the contribution to $U(P, f)$ from these intervals is at most $\frac{\epsilon}{2}$, and the contribution of the other intervals is at most $\frac{\epsilon}{2}$ because $f$ is bounded by $\frac{\epsilon}{2}$ there and the total size of these intervals is at most 1. We therefore have $U(P, f) < \epsilon$ as required.