HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 9: DUE THURSDAY 19 APRIL, AT 11:00 IN 2-108.

(1) (10 points) Show that if $f : [a,b] \longrightarrow \mathbb{R}^k$ is $(n+1)$ times differentiable, and $x, y \in [a,b]$, then

$$f(y) = f(x) + f'(x)(y-x) + \cdots + \frac{f^{(n)}(x)}{n!}(y-x)^n + E(y)$$

where

$$|E(y)| \leq sup_{[a,b]}|f^{(n+1)}|\frac{|y-x|^{n+1}}{(n+1)!}$$

Hint: try taking the dot product of $E(t)$ with $E(y)$ to get an $\mathbb{R}$ valued function.

(2) (10 points) Show that if $f : [a,b] \longrightarrow \mathbb{R}$ is differentiable and there exists some c so that

$$|f'(x)| \leq c|f(x)|$$

then if $f(a) = 0$, $f(x)$ must be 0 for all $x \in [a,b]$.

Hint: don't try to integrate this as f' is not necessarily Riemann integrable...Try to think what the mean value Theorem can tell us about the relationship between the supremum of f and f' on some small neighborhood of a.

(3) (a) (10 points) Suppose that f and g are real differentiable functions on an interval $[a,b]$ that satisfy the differential equation

$$f'(t) = H(t, f(t))$$

$$g'(t) = H(t, g(t))$$

Where $H : [a,b] \times R \longrightarrow \mathbb{R}$ satisfies the following condition. There exists some constant $c \in \mathbb{R}$ so that

$$|H(t, s_1) - H(t, s_2)| \leq c|s_1 - s_2|$$

Show that if $f(a) = g(a)$ then f and g are equal everywhere.

Hint: apply the previous problem to $f - g$.

(b) (5 points) Show that if $H(t,s) = |s|^{\frac{1}{2}}$ that this is not true.

(4) Denote the space of all continuous real valued functions on $[a,b]$ by $C([a,b])$.

(a) (10 points) Suppose that α is a strictly increasing function on $[a,b]$. (In other words $\alpha(x) > \alpha(y)$ if $x > y$.) Define the following inner product for $f, g \in C([a,b])$:

$$\langle f, g \rangle := \int_a^b fg d\alpha$$

Define the norm of $f \in C([a,b])$ to be

$$\|f\|_2 := (\langle f, f\rangle)^{\frac{1}{2}}$$

Prove the following inequality (called the Cauchy-Schwartz inequality):

$$\langle f, g\rangle \leq \|f\|_2 \|g\|_2$$

(b) (10 points) Show that the following defines a metric on $C([a,b])$:

$$d(f,g) := \|f-g\|_2$$

(c) (5 points) Give an example showing that this is not a metric on $\mathcal{R}(\alpha)$.

(5) (10 points) Recall that any rational in $\mathbb{Q}$ has a unique reduced form $\frac{p}{q}$ where p and q are integers with no common factors and $q > 0$. Define the function f by $f(x) = \frac{1}{q}$ if $x = \frac{p}{q}$ and $f(x) = 0$ for $x \notin \mathbb{Q}$. Prove directly that f is Riemann integrable on $[0,1]$ and find

$$\int_0^1 f dx$$

The following problem is not to be handed in: Define the following for any $f \in \mathcal{R}(\alpha)$ on the interval $[a,b]$ and $q \in (0,\infty)$:

$$\|f\|_q = \left(\int_a^b |f|^q\right)^{\frac{1}{q}}$$

Prove that if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

This is called Hölder's inequality. (For a hint see problem 10 in Rudin.) Prove also that the following gives a metric on $C([a,b])$ in the case that α is a strictly increasing function:

$$d(f,g) = \|f-g\|_q$$