HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007
ASSIGNMENT 6: DUE THURSDAY 22 MARCH, AT 11:00 IN 2-108.

(1) (10 points) Show that the Cauchy product of two absolutely convergent series is absolutely convergent.

Let $\sum a_n$ and $\sum b_n$ be absolutely convergent. We must show that the series defined by their Cauchy product $\sum c_n$ is absolutely convergent. Here

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

By repeated use of the triangle inequality we have that

$$|c_n| \leq \sum_{i=0}^{n} |a_i||b_{n-i}|$$

We know that the Cauchy product of a convergent series with an absolutely convergent series converges. As both $\sum|a_i|$ and $\sum|b_i|$ are absolutely convergent series, their Cauchy product converges. The terms in the Cauchy product are the right hand side of the above inequality, so the comparison test tells us that $\sum|c_n|$ converges, so $\sum c_n$ is absolutely convergent.

(2) (20 points) Let the partial sums of a series be given by

$$s_n := \sum_{i=0}^{n} c_n$$

and define a new sequence given by their average as follows:

$$\sigma_n := \frac{\sum_{i=0}^{n} s_n}{n+1}$$

(a) Prove that the sequence σ_n converges to $\sum c_n$ if $\sum c_n$ converges.

Suppose that

$$\sum c_n = S$$

Then $s_n \to S$, so s_n is bounded, so there exists some M so that $|s_n - S| \leq M$. Also, for any $\epsilon > 0$, there exists an N so that whenever $n > N$, $|s_n - S| < \frac{\epsilon}{2}$. Now suppose that $n > N + \frac{2MN}{\epsilon}$.

$$|\sigma_n - S| = \left| \sum_{i=0}^{n} \frac{s_i - S}{n+1} \right| \leq \sum_{i=0}^{N} \frac{|s_i - S|}{n+1} + \sum_{i=N+1}^{n} \frac{|s_i - S|}{n+1} < \epsilon$$

So σ_n converges to S.

(b) Give an example of a series which doesn't converge for which σ_n converges.

Consider the series $\sum_{n=0}^{\infty}(-1)^n$. This series does not converge as the size of the terms does not converge to 0. For this series

$$s_n = \frac{1+(-1)^n}{2}$$

$$\sigma_n = \frac{1}{2} + \frac{1+(-1)^n}{2(n+1)}$$

The sequence σ_n converges to $\frac{1}{2}$.

(3) (25 points) This question will construct the completion X^* of a metric space X.

(a) Show that if p_n and q_n are Cauchy sequences, then $d(p_n, q_n)$ is a convergent sequence.

If $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences, then for any $\epsilon > 0$, there exists some N_1 and N_2 so that if $m, n > N_1$, then $d(p_n, p_m) < \frac{\epsilon}{2}$, and if $m, n > N_2$, $d(q_n, q_m) < \frac{\epsilon}{2}$. Then we have that if $n, m > \max(N_1, N_2)$,

$$|d(p_n, q_n) - d(p_n, q_m)| < \frac{\epsilon}{2}$$

$$|d(p_n, q_m) - d(p_m, q_m)| < \frac{\epsilon}{2}$$

Therefore,

$$|d(p_n, q_n) - d(p_m, q_m)| < \epsilon$$

This means that $d(p_n, q_n)$ forms a Cauchy sequence in $\mathbb{R}$. As $\mathbb{R}$ is complete, this Cauchy sequence must converge.

(b) Show that the following is an equivalence relation on the set of Cauchy sequences in X: The Cauchy sequence $\{p_n\}$ is equivalent to $\{q_n\}$ if

$$\lim_{n\to\infty} d(p_n, q_n) = 0$$

To show that this is an equivalence relation, we must show that it is reflexive, symmetric and transitive. $d(p_n, p_n) = 0$, so a Cauchy sequence is always equivalent to itself, and this relation is reflexive.

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n)$$

so a sequence $\{p_n\}$ is equivalent to $\{q_n\}$ if $\{q_n\}$ is equivalent to $\{p_n\}$ and this relation is symmetric.

$$\lim_{n\to\infty} d(p_n, r_n) \leq \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n)$$

So if $\{p_n\}$ is equivalent to $\{q_n\}$ and $\{q_n\}$ is equivalent to $\{r_n\}$, then $\{p_n\}$ is equivalent to $\{r_n\}$. In otherwords, this relation is transitive.

(c) Let X^* be the set of equivalence classes of Cauchy sequences in X. Show that the following defines a metric on X^*: Given $P \in X^*$ and $Q \in X^*$, let $\{p_n\}$ and $\{q_n\}$ be Cauchy sequences in the equivalence classes P and Q respectively. Define the distance between P and Q to be

$$d(P,Q) := \lim_{n\to\infty} d(p_n, q_n)$$

We must show that $d(P,Q)$ is well defined, and that it is a metric. First, lets show that it is well defined. Let $\{p_n\}$ and $\{p'_n\}$ be two equivalent Cauchy sequences. The triangle inequality tells us that

$$|d(p_n, q_n) - d(p'_n, q_n)| \leq d(p_n, p'_n)$$

Therefore, we have that

$$|\lim_{n\to\infty} d(p_n, q_n) - \lim_{n\to\infty} d(p'_n, q_n)| \leq \lim_{n\to\infty} d(p_n, p'_n) = 0$$

Therefore $d(P,Q)$ does not depend on the choice of Cauchy sequence representing P, and similarly it does not depend on the choice of Cauchy sequence representing Q, so it is well defined.
We have defined $d(P,Q)$ so that it is 0 if and only if the equivalence classes P and Q are equal. Otherwise, it is positive, as it is the limit of non negative numbers. Clearly $d(P,Q) = d(Q,P)$, so d is symmetric. The triangle inequality follows from the following:

$$\lim_{n\to\infty} d(p_n, r_n) \leq \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n)$$

(d) Prove that X^* is complete with this metric.

We must show that any Cauchy sequence $\{P_n\}$ converges to some limit P in X^*. We do this as follows: First, each P_n is an equivalence class of Cauchy sequences, to make our life easier, we will choose a Cauchy sequence $\{p_{n,k}\}$ in each P_n. We must construct a Cauchy sequence $\{p_n\}$ in X who's equivalence class is the limit of these P_n. For each n, there exists some k so that for all $l, m \geq k$, $d(p_{n,l}, p_{n,m}) < \frac{1}{n}$. Choose the nth term of our sequence $p_n = p_{n,k}$.
Lets show that this is a Cauchy sequence. Given any $\epsilon > 0$, choose $N > \frac{4}{\epsilon}$ so that for all $m, n > N$, $d(P_n, P_m) < \frac{\epsilon}{2}$. We then have for all $m, n > N$

$$d(p_n, p_m) \leq d(p_n, p_{n,l}) + d(p_m, p_{m,l}) + d(p_{n,l}, p_{m,l})$$

Taking the limit as $l \to \infty$ gives

$$d(p_n, p_m) < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon$$

so $\{p_n\}$ is a Cauchy sequence. Let P be the equivalence class containing $\{p_n\}$. We must now show that $P_n \to P$.

$$\begin{aligned} d(P_n, P) &= \lim_{m\to\infty} d(p_{n,m}, p_m) \\ &\leq \lim_{m\to\infty} \left(d(p_{n,m}, p_n) + d(p_n, p_m)\right) \\ &< \frac{1}{n} + \lim_{m\to\infty} d(p_n, p_m) \end{aligned}$$

As $\{p_n\}$ is a Cauchy sequence, we can choose $N > \frac{2}{\epsilon}$ so that if $m, n > N$ then $d(p_n, p_m) < \frac{\epsilon}{2}$. This means that if $n > N$ $d(P_n, P) < \epsilon$ so $P_n \to P$ as required.

(e) Consider X to be a subset of X^* by sending $x \in X$ to the equivalence class $P_x \in X^*$ containing the constant Cauchy sequence with every member equal to x. Prove that

$$d(P_x, P_y) = d(x, y)$$

$$d(P_x, P_y) = \lim_{n\to\infty} d(x, y) = d(x, y)$$

This means that we can consider X as a subset of X^*.

(4) (10 points) Show that the completion of the rational numbers is the real numbers. (The operations of addition and multiplication on the completion of $\mathbb{Q}$ comes from adding and multiplying Cauchy sequences.)

Note that $\mathbb{R}$ is complete and contains $\mathbb{Q}$ as a dense subset. Every Cauchy sequence $\{q_n\}$ inside $\mathbb{Q}$ has a limit $Q \in \mathbb{R}$. If $\{q_n\}$ and $\{p_n\}$ are two Cauchy sequences with limits Q and P, then

$$|P - Q| = \lim_{n\to\infty} |p_n - q_n|$$

This is the same metric as on the completion of $\mathbb{Q}$, so $\mathbb{R}$ must contain the completion of $\mathbb{Q}$. The fact that $\mathbb{Q}$ is dense implies that for any point $P \in \mathbb{R}$ there exists some sequence $\{p_n\}$ inside $\mathbb{Q}$ converging to it. Any convergent sequence is a Cauchy sequence, so $\{p_n\}$ is a Cauchy sequence in $\mathbb{Q}$ which has limit P. Therefore, the completion of $\mathbb{Q}$ is equal to $\mathbb{R}$ with its usual metric.

(5) (10 points) Show that (Y, d) is complete if and only if for every metric space (X, d) which contains it, Y is a closed subset of X.

If Y is a complete subset of X, then if $p \notin Y$, then there exists some distance $\epsilon > 0$ so that $B_\epsilon(p) \cap Y = \emptyset$. (If this was not the case, then there would exist some sequence of points $p_i \in Y$ so that $d(p_i, p) < \frac{1}{i}$. Then $p_i \to p$ must be a Cauchy sequence, which therefore must have a limit inside Y, but $p \notin Y$, a contradiction.) The complement of Y is therefore open, so Y is a closed subset of X.

Now suppose that Y is not complete. We must show that there exists some metric space X containing Y so that Y is not a closed subset of X. We will take X to be the completion of Y. This contains Y, and is complete.

As Y is not complete, there exists some Cauchy sequence $\{p_i\}$ inside Y which is not convergent inside Y. As X is complete, $\{p_i\}$ must converge to some point $p \in X$ so that $p \notin Y$. Any sequence inside a closed set which converges has its limit inside that set. Therefore Y can not be closed.

The following question is not to be handed in:

(1) Show that a metric space X is compact if every sequence in X has a convergent subsequence as follows:
   (a) Show that for any $\epsilon > 0$, there exists some finite number N so that there are N balls of radius ϵ which cover X. (Show that if this was not true, then there would be an infinite number of balls of radius $\frac{\epsilon}{2}$ which would not intersect each other, and therefore a sequence with no convergent subsequence.)
   (b) Show that if $\{U_\alpha\}$ is an open cover of X with no finite subcover, there must be a sequence $\{p_n\}$ so that $B_{\frac{1}{n}}(p_n)$ has no finite subcover. Show that the fact that this sequence has a convergent subsequence will lead to a contradiction.