HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 6: DUE THURSDAY 22 MARCH, AT 11:00 IN 2-108.

(1) (10 points) Show that the Cauchy product of two absolutely convergent series is absolutely convergent.

(2) (20 points) Let the partial sums of a series be given by

$$s_n := \sum_{i=0}^{n} c_n$$

and define a new sequence given by their average as follows:

$$\sigma_n := \frac{\sum_{i=0}^{n} s_n}{n+1}$$

(a) Prove that the sequence σ_n converges to $\sum c_n$ if $\sum c_n$ converges.

(b) Give an example of a series which doesn't converge for which σ_n converges.

(3) (25 points) This question will construct the completion X^* of a metric space X.

(a) Show that if p_n and q_n are Cauchy sequences, then $d(p_n, q_n)$ is a convergent sequence.

(b) Show that the following is an equivalence relation on the set of Cauchy sequences in X: The Cauchy sequence $\{p_n\}$ is equivalent to $\{q_n\}$ if

$$\lim_{n\to\infty} d(p_n, q_n) = 0$$

(c) Let X^* be the set of equivalence classes of Cauchy sequences in X. Show that the following defines a metric on X^*: Given $P \in X^*$ and $Q \in X^*$, let $\{p_n\}$ and $\{q_n\}$ be Cauchy sequences in the equivalence classes P and Q respectively. Define the distance between P and Q to be

$$d(P, Q) := \lim_{n\to\infty} d(p_n, q_n)$$

(d) Prove that X^* is complete with this metric.

(e) Consider X to be a subset of X^* by sending $x \in X$ to the equivalence class $P_x \in X^*$ containing the constant Cauchy sequence with every member equal to x. Prove that

$$d(P_x, P_y) = d(x, y)$$

(4) (10 points) Show that the completion of the rational numbers is the real numbers. (The operations of addition and multiplication on the completion of $\mathbb{Q}$ comes from adding and multiplying Cauchy sequences.)

(5) (10 points) Show that (Y, d) is complete if and only if for every metric space (X, d) which contains it, Y is a closed subset of X.

The following question is not to be handed in:

(1) Show that a metric space X is compact if every sequence in X has a convergent subsequence as follows:
   (a) Show that for any $\epsilon > 0$, there exists some finite number N so that there are N balls of radius ϵ which cover X. (Show that if this was not true, then there would be an infinite number of balls of radius $\frac{\epsilon}{2}$ which would not intersect each other, and therefore a sequence with no convergent subsequence.)
   (b) Show that if $\{U_\alpha\}$ is an open cover of X with no finite subcover, there must be a sequence $\{p_n\}$ so that $B_{\frac{1}{n}}(p_n)$ has no finite subcover. Show that the fact that this sequence has a convergent subsequence will lead to a contradiction.