HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007
ASSIGNMENT 11: DUE THURSDAY 10 MAY, AT 11:00 IN 2-108.

(1) Let $\{f_n\}$ be an equicontinuous sequence of functions $f_n : K \longrightarrow \mathbb{C}$ defined on a compact metric space K. Prove that if $\{f_n\}$ converges pointwise, then it must converge uniformly.

If $\{f_n\}$ converges pointwise, $\{f_n\}$ must be pointwise bounded. Our sequence $\{f_n\}$ is therefore a pointwise bounded and equicontinuous sequence of functions defined on a compact set. We therefore know that there must be a subsequence that converges uniformly to some continuous f. As limits are unique, f_n must also converge pointwise to f. Any sequence of continuous functions which converge pointwise to a continuous function on a compact set must also converge uniformly.

(2) Suppose that $\phi : \mathbb{R} \longrightarrow [0, \infty)$ is a smooth (which means that all derivatives of ϕ exist) positive 'bump' function so that

$$\phi(x) = 0 \text{ for all } |x| \geq 1$$

$$\int_{-1}^{1} \phi(x)dx = 1$$

Suppose that f is some function that is Riemann integrable on any finite interval. (In other words the integral of f exists on $[a, b]$ for every finite interval $[a, b]$.) Define the convolution of ϕ with f to be

$$(\phi * f)(t) := \int_{-1}^{1} \phi(x)f(t + x)dx$$

(a) Prove that $\phi * f$ is continuous.
(Hint: use that ϕ is uniformly continuous and the observation that $(\phi * f)(t) = \int \phi(x - t)f(x)dx$.)

First, note that as ϕ is constant everywhere outside $(-1, 1)$, and any continuous function restricted to the compact set $[-1, 1]$ is uniformly continuous, ϕ must be uniformly continuous. Therefore for all $\epsilon > 0$, there exists some $\delta > 0$ so that if $|x - y| < \delta$, $|\phi(x) - \phi(y)| < \epsilon$. Let $t_1 < t_2 < t_1 + \delta$.

$$|\phi * f(t_1) - \phi * f(t_2)| = \left| \int_{t_1-1}^{t_2+1} (\phi(x - t_1) - \phi(x - t_2))f(x)dx \right|$$
$$< \epsilon \int_{t_1-1}^{t_2+1} |f(x)|dx$$

Therefore, if $|s-t|<\delta$,

$$|\phi * f(s) - \phi * f(t)| < \epsilon \int_{s-\delta-1}^{s+\delta+1} |f| dx$$

This implies that $\phi * f$ is continuous.

(b) Prove that $\phi * f$ is differentiable, and find an expression for its derivative. Show that if f is uniformly bounded then the derivative of $\phi * f$ is uniformly bounded.
(Hint: show somehow that $\frac{(\phi(x+h)-\phi(x))}{h}$ converges uniformly to $\phi'(x)$ as $h \to 0$, then use what you know about integration and uniform convergence.)

First, note that as ϕ is twice differentiable, $\phi'(x)$ is continuous. As $\phi'(x)$ is 0 outside the compact set $[-1,1]$, it is also uniformly continuous. Therefore, for any $\epsilon > 0$, there exists a $\delta > 0$ so that if $|x-y|<\delta$, $|\phi'(x)-\phi'(y)|<\epsilon$. Now choose $h<\delta$. The mean value theorem tells us that for any x, there exists some x' between x and $x+h$ so that $\frac{\phi(x+h)-\phi(x)}{h} = \phi'(x')$. The fact that $|x-x'|<\delta$ then tells us that

$$\left| \frac{\phi(x+h)-\phi(x)}{h} - \phi'(x) \right| = |\phi'(x') - \phi'(x)| < \epsilon \ \forall h < \delta$$

This tells us that $\frac{(\phi(x+h)-\phi(x))}{h}$ converges uniformly to $\phi'(x)$ as $h \to 0$. Now

$$\begin{aligned}
(\phi * f)'(t) :&= \lim_{h\to 0} \frac{\phi * f(t+h) - \phi * f(t)}{h} \\
&= \lim_{h\to 0} \int_{t-2}^{t+2} \frac{\phi(x-t-h)-\phi(x-t)}{h} f(x) dx \\
&= \int_{t-2}^{t+2} -\phi'(x-t) f(x) dx = -\phi' * f
\end{aligned}$$

The last line follows because f is bounded on the interval $[t-2, t+2]$ so on this interval, the integrand converges uniformly.

(c) Define

$$\phi_n(x) := n\phi(nx)$$

Prove that if f is continuous at t, then $(\phi_n * f)(t)$ converges to $f(t)$. Prove that if f is uniformly continuous, then the sequence $\phi_n * f$ converges uniformly to f.

Suppose that f is continuous at t. Therefore, for all $\epsilon > 0$, there exists some $\delta > 0$ so that if $|x-t|<\delta$, $|f(x)-f(t)|<\epsilon$. Now for any $n > \frac{1}{\delta}$, noting that the integral of ϕ_n is 1 and ϕ_n vanishes outside of $[-\frac{1}{n}, \frac{1}{n}]$, we have

$$|(\phi_n * f)(t) - f(t)| = \left|\int_{-\frac{1}{n}}^{\frac{1}{n}} \phi_n(x)(f(x-t) - f(t))dx\right|$$
$$\leq \int_{-\frac{1}{n}}^{\frac{1}{n}} \phi_n(x)|f(x-t) - f(t)|dx < \epsilon$$

Therefore, $(\phi * f)(t)$ converges to $f(t)$.

In the case that f is uniformly continuous, for any $\epsilon > 0$, there exists some $\delta > 0$ so that if $|t - x| < \delta$, $|f(t) - f(x)| < \epsilon$. The above calculation then shows that if $n > \frac{1}{\delta}$, $|f(t) - (\phi_n * f)(t)| < \epsilon$ for all t. In other words, $\phi_n * f$ converges to f uniformly.

(d) Prove that if f is not continuous, then $\phi_n * f$ can not converge to f uniformly.

Note that $\phi_n * f$ is a sequence of continuous functions. If $\phi_n * f$ converged uniformly to f, f would have to be continuous.

(3) Suppose that $\phi_n : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a sequence of continuous functions that converges uniformly to ϕ. Suppose also that $f_n : [0,1] \longrightarrow \mathbb{R}$ is a uniformly bounded sequence of differentiable functions that satisfy the differential equation

$$f_n'(t) = \phi_n(t, f_n(t))$$

(a) Show that the sequence $\{f_n\}$ is equicontinuous. (Hint: We are just interested in ϕ_n restricted to a bounded subset of $\mathbb{R}^2$...why must ϕ_n be bounded there?)

Choose M so that $|f_n(x)| < M$ for all x and n. We can just consider ϕ_n to be a sequence of functions on the compact set $[0,1] \times [-M, M]$. On this compact set, ϕ and each ϕ_n must be bounded because they are contiunous. As ϕ_n converges to ϕ, these functions must be uniformly bounded, so there exists some M' so that $|\phi_n(x,y)| < M'$ for all n and $x \in [0,1]$, $y \in [-M, M]$. We therefore have that $|f_n'| < M'$. Uniformly bounded derivatives implies equicontinuity...in particular, the Mean Value Theorem implies that if $|x - y| < \frac{\epsilon}{M'}$, $|f(x) - f(y)| < \epsilon$.

(b) Show that the sequence $\{f_n'\}$ is equicontinuous. (Hint: show first that $\{\phi_n\}$ is equicontinuous restricted to a bounded subset of $\mathbb{R}^2$, and then use that to prove that f_n' must be equicontinuous.)

$\{\phi_n\}$ is a sequence of continuous functions which converge uniformly to ϕ, therefore restricted to any compact set, Theorem 7.24 in Rudin tells us that $\{\phi_n\}$ must be equicontinuous. In particular, $\{\phi_n\}$ is equicontinuous when restricted to $[0,1] \times [-M, M]$. Therefore, as f_n is equicontinuous, $\phi_n(x, f_n(x))$ is equicontinuous, so f_n' is equicontinuous as required.

(c) Show that there exists a subsequence of these that converge uniformly to a continuously differentiable function $f : [0,1] \longrightarrow \mathbb{R}$ satisfying

$$f'(t) = \phi(t, f(t))$$

We have shown so far that $\{f_n'\}$ is equicontinous and bounded, (and is defined on the compact interval $[0,1]$), therefore we know that there exists a convergent subsequence. As $\{f_n\}$ is also equicontinuous and bounded there exists a subsequence of this so that both f_{n_i} and f_{n_i}' converge uniformly. This subsequence $\{f_{n_i}\}$ therefore converges to some continuously differentiable function f. We have that

$$f'(x) = \lim_{i\to\infty} f_{n_i}'(x) = \lim \phi_{n_i}(x, f_{n_i}(x)) = \phi(x, f(x))$$