HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007 ASSIGNMENT 11: DUE THURSDAY 10 APRIL, AT 11:00 IN 2-108.

(1) Let $\{f_n\}$ be an equicontinuous sequence of functions $f_n : K \longrightarrow \mathbb{C}$ defined on a compact metric space K. Prove that if $\{f_n\}$ converges pointwise, then it must converge uniformly.

(2) Suppose that $\phi : \mathbb{R} \longrightarrow [0, \infty)$ is a smooth (which means that all derivatives of ϕ exist) positive 'bump' function so that

$$\phi(x) = 0 \text{ for all } |x| \geq 1$$

$$\int_{-1}^{1} \phi(x)dx = 1$$

Suppose that f is some function that is Riemann integrable on any finite interval. (In other words the integral of f exists on $[a, b]$ for every finite interval $[a, b]$.) Define the convolution of ϕ with f to be

$$(\phi * f)(t) := \int_{-1}^{1} \phi(x)f(t + x)dx$$

(a) Prove that $\phi * f$ is continuous.
(Hint: use that ϕ is uniformly continuous and the observation that $(\phi * f)(t) = \int \phi(x - t)f(x)dx$.)

(b) Prove that $\phi * f$ is differentiable, and find an expression for its derivative. Show that if f is uniformly bounded then the derivative of $\phi * f$ is uniformly bounded.
(Hint: show somehow that $\frac{(\phi(x+h)-\phi(x))}{h}$ converges uniformly to $\phi'(x)$ as $h \to 0$, then use what you know about integration and uniform convergence.)

(c) Define

$$\phi_n(x) := n\phi(nx)$$

Prove that if f is continuous at t, then $(\phi_n * f)(t)$ converges to $f(t)$. Prove that if f is uniformly continuous, then the sequence $\phi_n * f$ converges uniformly to f.

(d) Prove that if f is not continuous, then $\phi_n * f$ can not converge to f uniformly.

(3) Suppose that $\phi_n : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a sequence of continuous functions that converges uniformly to ϕ. Suppose also that $f_n : [0, 1] \longrightarrow \mathbb{R}$ is a uniformly bounded sequence of differentiable functions that satisfy the differential equation

$$f_n'(t) = \phi_n(t, f_n(t))$$

(a) Show that the sequence $\{f_n\}$ is equicontinuous. (Hint: We are just interested in ϕ_n restricted to a bounded subset of $\mathbb{R}^2$...why must ϕ_n be bounded there?)

(b) Show that the sequence $\{f_n'\}$ is equicontinuous. (Hint: show first that $\{\phi_n\}$ is equicontinuous restricted to a bounded subset of $\mathbb{R}^2$, and then use that to prove that f_n' must be equicontinuous.)

(c) Show that there exists a subsequence of these that converge uniformly to a continuously differentiable function $f : [0,1] \longrightarrow \mathbb{R}$ satisfying

$$f'(t) = \phi(t, f(t))$$