HOMEWORK FOR 18.100B AND 18.100C, SPRING 2007
ASSIGNMENT 10. Solutions.

1. Assume f is a real, differentiable function with continuous derivative on $[a, b]$, $f(a) = f(b) = 0$, and $\int_a^b f^2(x)\, dx = 1$. Prove that

(1) $\int_a^b xf(x)f'(x)\, dx = -\frac{1}{2}$, and
(2) $\int_a^b [f'(x)]^2\, dx \cdot \int_a^b x^2 f^2(x)\, dx > \frac{1}{4}$.

Solution. (a) It follows from integration by parts: $\int_a^b xf(x)f'(x)\, dx = \frac{1}{2}xf^2(x)|_a^b - \frac{1}{2}\int_a^b f^2(x)\, dx$.
(b) Apply the Schwarz inequality and part (a).

2. Consider the series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}, \quad x > 0.$$

(1) On what intervals does the series converges uniformly? On what intervals does it fail to converge uniformly?
(2) Is f continuous wherever the series converges?

Solution. (a) If $x > 0$, $\sum_{n=1}^{\infty} \frac{1}{1+n^2x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{1}{n^2+\frac{1}{x}}$. Since $\frac{1}{n^2+\frac{1}{x}} < \frac{1}{n^2}$, and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, the series $\sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ converges (in fact absolutely, as it has positive terms anyway).
If $x \geq a > 0$, then $\frac{1}{1+n^2x} \leq \frac{1}{1+n^2a}$. Since $\sum \frac{1}{1+n^2a}$ converges, by Weierstrass' M-test, $\sum \frac{1}{1+n^2x}$ converges unformly on every interval $[a, b] \subset (0, \infty)$.
On intervals of the form $(0, b]$ the series does not converge uniformly. Assume by way of contradiction that it does. Then Cauchy's test implies that there exists $N > 0$ such that $\sum_{n=N}^{m}\frac{1}{1+n^2x} < \frac{1}{2}$, for all $x \in (0, b]$. But if we set $x = \frac{1}{N^2}$ we get a contradiction.
(b) Since all the terms of the series are continuous functions whenever they are defined, $f(x)$ is continuous on all intervals where it converges uniformly. Every point of convergence (see (a)) can be put in an interval $[a, b]$ where the series converges uniformly. Therefore, f is continuous for all values of x for which is converges.

3. Define $f_n(x) = \frac{x}{1+nx^2}$, for $n \geq 1$. Prove that:

(1) $\{f_n\}$ converges uniformly on $\mathbb{R}$ to some function f, and
(2) $\{f_n'(x)\}$ converges pointwise to $\{f'(x)\}$ for all $x \neq 0$, but not for $x = 0$.

Solution. (a) Of course, as the denominator of f_n is strictly positive, f_n are defined everywhere, and are continuously differentiable.It is clear that $\{f_n(x)\}$ converges pointwise to $f(x) = 0$ for every x. To check uniform convergence, let $\epsilon > 0$ be given. Note that

$$f_n'(x) = \frac{(1+nx^2) - 2nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

In particular, $f_n(x)$ has a local maximum at $x = 1/\sqrt{n}$ and a local minimum at $x = -1/\sqrt{n}$, and it's clear that these are actually a global maximum and minimum. So for any x,

$$\begin{aligned} |f_n(x)| &\leq \max\{|f_n(1/\sqrt{n})|, |f_n(-1/\sqrt{n})|\} \\ &\leq \frac{1}{2\sqrt{n}} \end{aligned}$$

So for any $\epsilon > 0$, if $N > 1/4\epsilon^2$, we will have $|f_n(x) - f(x)| < \epsilon$ for every $x \in \mathbb{R}$ and $n > N$. So, the $\{f_n\}$ converge uniformly to the function $f(x) = 0$ as claimed.

(b) Then, for any $x \neq 0$,

$$\lim_{n\to\infty} f_n'(x) = 0$$

as the n^2 in the denominator will dominate all the other terms. Thus $f'(x) = 0 = \lim_{n\to\infty} f_n'(x)$. However, at 0, we find $f_n'(0) = 1$ for any n, so $\lim_{n\to\infty} f_n'(x) = 1$, which is different from $f'(0) = 0$.

In particular, this shows the necessity of the "uniform convergence of f_n'" condition in Rudin Theorem 7.17.

4. Recall the step function

$$I(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}.$$

Let $\{x_n\}$ be a sequence of distinct points in the interval (a, b), and $\sum c_n$ and absolutely convergent series. Prove that the series of functions

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n), \quad x \in [a, b]$$

converges uniformly.

In addition, show that f is continuous at every $x \neq x_n$.

Solution. Since $I(x - x_n) \in \{0, 1\}$, for all x, and $\sum |c_n|$ converges, by the Weierstrass M-test, the series $\sum_{n=1}^{\infty} c_n I(x - x_n)$ converges uniformly and absolutely. If $x \notin \{x_1, x_2, \dots\}$, then I is continuous at $x - x_n$ for every n. Therefore this is a uniformly convergent series of continuous functions at x, and so it is continuous.

Assume $x = x_m$ for some m. There exists an open interval V around x_m, such that there no x_i, $i \neq m$, are in this interval. Let s, t be arbitrary points in V with $t < x_m < s$. Since the series is absolutely convergent, $f(s) - f(t) = \sum c_n[I(s - x_n) - I(t - x_n)]$. Clearly s and t are in the same side of x_n, for every $n \neq m$, which means that $I(s - x_n) - I(t - x_n) = 0$, for

$n \neq m$. Then $f(s) - f(t) = c_m[I(s - x_m) - I(t - x_m)] = c_m$, which means that f is not continuous at x_m.

5. Put $P_0 = 0$, and define, for $n \geq 0$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that the sequence of polynomials $\{P_n(x)\}_n$ converges uniformly to $|x|$ on the interval $[-1, 1]$. (Hint: see ex. 23/p. 169.)

Solution. From the recursion formula for P_n, it follows immediately that

$$|x| - P_{n+1}(x) = (|x| - P_n(x))\left(1 - \frac{|x| + P_n(x)}{2}\right). \tag{1}$$

$P_0 = 0$. Assume by induction that $0 \leq P_n(x) \leq |x|$, for all $x \in [-1, 1]$. In (1), the right hand side is nonnegative then, and so $P_{n+1}(x) \leq |x|$. From the recursion formula, we also have $P_{n+1} \geq 0$.

Again from the recursion formula, $P_{n+1}(x) - P_n(x) = \frac{1}{2}(x^2 - P_n^2(x)) \geq 0$.

So we have proved by induction that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$, for all n and all $x \in [-1, 1]$.

By induction, assume $0 \leq |x| - P_n(x) \leq |x|\left(1 - \frac{|x|}{2}\right)^n$. (Note that this is clearly true for $n = 0$.) Then from formula (1), using $P_n(x) \geq 0$ and the induction hypothesis, $|x| - P_{n+1}(x) \leq (|x| - P_n(x))\left(1 - \frac{|x|}{2}\right) \leq |x|\left(1 - \frac{|x|}{2}\right)^{n+1}$.

Therefore, we have proved $0 \leq |x| - P_n(x) \leq |x|\left(1 - \frac{|x|}{2}\right)^n$. Consider the function $g(y) = y(1 - \frac{y}{2})^n$, for $0 \leq y \leq 1$. By computing the derivative (and checking endpoints), we find that the maximum of this function is achieved at $y = \frac{2}{n+1}$. So $g(y) \leq \frac{2}{n+1}(1 - \frac{1}{n+1})^n \leq \frac{2}{n+1}$. Set $y = |x|$, and we find that

$$0 \leq |x| - P_n(x) \leq \frac{2}{n+1}.$$

Fix $\epsilon > 0$, and let $N > 0$ be greater than $\frac{2}{\epsilon} - 1$. Then for all $n \geq N$, and all $x \in [-1, 1]$, $0 \leq |x| - P_n(x) < \epsilon$. This proves that $\{P_n(x)\}$ converges uniformly on $[-1, 1]$ to $|x|$.