1 Chebyshev polynomials

Recall the trigonometric definition of the Chebyshev polynomials:

\[ T_n(x) = \cos(n\theta), \quad \theta = \cos^{-1}(x). \quad (1.1) \]

a) Prove that \( T_n(x) \) is a polynomial of degree \( n \) in \( x \). \textit{Hint:} use the cosine addition formulae for \( \cos((n+1)\theta) \) and \( \cos((n-1)\theta) \) to find a recurrence relation for \( T_{n+1}(x) \), and prove the statement by induction starting with \( n = 0, 1 \). That is, show that \( \deg(T_0) = 0 \), that \( \deg(T_1) = 1 \), and that \( \deg(T_{n+1}) = \deg(T_n) + 1 \).

b) Prove that the Chebyshev polynomials are orthogonal under the weight \( w(x) = (1 - x^2)^{-1/2} \). \textit{Hint:} start the orthogonality of cosine modes on \([0, \pi]\):

\[ \int_0^\pi \cos(m\theta) \cos(n\theta) d\theta \propto \delta_{n,m} \quad (1.2) \]

c) The \textit{second-kind} Chebyshev polynomials \( U_n(x) \) are related to the derivatives of the regular Chebyshev polynomials as \( \partial_x T_n(x) = nU_{n-1}(x) \). Show that \( \{U_n(x)\} \) are also a set of orthogonal polynomials. What is the weight function under which they are orthogonal? \textit{Hint:} find a trigonometric definition for \( U_n(x) \) in terms of \( \theta \) and proceed similarly to part b.

\textit{Bonus:} I claim that the polynomials \( C_n^{(\alpha)}(x) \equiv \partial_x^{\alpha} T_n(x) \) are themselves a family of orthogonal polynomials for each \( \alpha \in \mathbb{N} \). This fact is key to producing banded discretizations of high-order PDEs. Based on the previous calculations, can you guess the form of the weight function \( w_\alpha(x) \) under which the \( C_n^{(\alpha)}(x) \) are orthogonal?


2 Chebyshev spectra

Here are four functions:

\[ f_1(x) = \cos(2\pi x) + \sin(2\pi x) \quad (2.1) \]
\[ f_2(x) = \cos(200\pi x) + \sin(200\pi x) \quad (2.2) \]
\[ f_3(x) = \sqrt{1 - x^2} \quad (2.3) \]
\[ f_4(x) = |x - 1/\pi|^3 \quad (2.4) \]

The numerically computed Chebyshev spectra up to degree 1000 of these functions on \( x \in [-1, 1] \) are plotted below. Match each function to one of the spectra, and briefly explain your choice based on the convergence theory for Chebyshev expansions and the function’s regularity.