18.095 Lecture 1

See slides for beginning of lecture.

Warmup—Drunkard's walk (described in slides)

Let $p(i)$ = prob get to $n$ before getting to 0 if start at $i$.

1. $p(0) = 0$
2. $p(n) = 1$
3. For $i = 1, 2, ..., n-1$,
   \[
   p(i) = \frac{1}{2} p(i+1) + \frac{1}{2} p(i-1)
   \]
   [Why?]

Can solve for $p(i)$, get $p(i) = \frac{i}{n}$ for $i = 0, 1, ..., n$

Circuit:

Can solve, get current of $\frac{1}{n}$ along path, voltage of $i^{th}$ vertex = $i/n$.

Same answer as above. Is this a coincidence?
Discrete Harmonic Functions

Have graph w/ 2 kinds of verts: boundary verts and internal verts. [Assume set of boundary verts ≠ Ø]

A function $f : \text{verts} \to \mathbb{R}$ is a (discrete) harmonic function if, for every internal node $x$,

$$f(x) = \sum_{y \in x} \frac{f(y) W_{xy}}{\text{deg}(x)}$$

where $\text{deg}(x) = \sum_{y \in x} W_{xy}$

Maximum Principle-Harmonic fn. on a connected graph achieves its max (and min) on boundary.

Proof- Suppose otherwise. Let $S =$ set of nodes achieving max $\Rightarrow S ≠ Ø$ (since contains all boundary verts)

∃ an edge $(xy)$ w/ $x \in S, y \notin S$ (since graph is connected) $f(x) > f(y)$, since $x$ achieves max & $y$ doesn't.

None of $x$'s nbrs have bigger val than $f(x)$, since $f(x)$ is max.

$\Rightarrow f(x)$ is (weighted) avg of numbers all $\leq f(x)$, with at least one strictly less than $f(x)$, which can't happen. This is a contradiction to assumption $f$ is harmonic. [Analogous argument works for min too]
Corollary [Uniqueness] - Suppose f, g harmonic functions and f(x) = g(x) for all boundary verts x. Then f(x) = g(x) for all x (i.e., for internal verts too).

Proof - Let h(x) = f(x) - g(x). Can check h is harmonic. h is 0 on all boundary verts \(\Rightarrow\) \(\max(h) = \min(h) = 0\) by maximum principle.

Voltage is harmonic - Say fix \(u(a) = 1\), \(u(b) = 0\). Let current flow, use Kirchhoff's laws to determine voltage @ all other verts.

Claim - V is harmonic (with boundary = \(\{a, b\}\))

Why? - For edge \((x, y)\), \(i_{xy} = \frac{u(x) - u(y)}{x_y} = (u(x) - u(y))w_{xy}\)

[Recall \(w_{xy} = \frac{1}{x_y}\)]

By fact that current in = current out for every internal vertex \(x\), \(0 = \sum_i i_{xy} = \sum_i (u(x) - u(y))w_{xy}\)

\[\Rightarrow u(x) = \sum_i u(y)w_{xy}\]

\[\Rightarrow u(x) = \sum_i u(y)w_{xy}\]

\[\sum_i w_{xy} = \deg(x)\]
Answer to walk question is harmonic.

Let $p(x)$ = prob. that walk starting at $x$ reaches $a$ before it reaches $b$.
If $\theta x$, prob. go to $y = \frac{wxy}{\deg(x)}$

Have $p(a) = 1$
$p(b) = 0$

$p(x) = \sum \frac{wxy}{\deg(x)} \cdot p(y)$

prob if $x \to y$
f in next step

So...

$p \& v$ harmonic w/ same bdry vals

$\Rightarrow p = v$

I.e. Prob. hit a before b when starting at $x$

Voltage if turn graph into circuit, the battery to set $V_a$. Voltage of $a=1$, voltage of $b=0$. 
If multiple absorbing states & want to know prob get to states 1, ..., k before states 1' ... k',
just set \( u(x) = 1 \) for states 1, ... , k, and \( u(x) = 0 \) for states 1' ... k'.

Why?

Know what voltage means. What about current?

Claim - \( I_{xy} = \text{net freq. walk that starts at } a \text{ goes through edge } (x,y) \text{ before getting to } b \). [Need to normalize correctly - see note @ end]

Why? Let \( u(x) = \text{avg. # times walk starting at } a \text{ visits } x \text{ before getting to } b \).

\( u(b) = 0 \)

For internal vertex \( x \), \( u(x) = \sum_y u(y) [\text{prob. } x \to y] \)

\( u \) not harmonic, but will show \( S(x) = u(x) \)

\( \deg(x) = \sum_y \frac{u(y) w_{xy}}{\deg(y)} \)

\( S(x) = \frac{1}{2} \sum_y \frac{u(y) \deg(y)}{\deg(x)} \)

\( S \) harmonic [not nec. = 1]

Let \( v_a = u(a) / \deg(a) = S(a) \)

By uniqueness, \( S = \text{voltages } v \text{ when set } u(a) = v_a \), \( v(b) = 0 \).
\[ i_{xy} = (v(x) - v(y)) \cdot w_{xy} = (s(x) - s(y)) \cdot w_{xy} \]

\[ = \left( \frac{u(x)}{\text{deg}(x)} - \frac{u(y)}{\text{deg}(y)} \right) \cdot w_{xy} \]

\[ = \frac{u(x) \cdot w_{xy}}{\text{deg}(x)} - \frac{u(y) \cdot w_{xy}}{\text{deg}(y)} \]

*avg # times take edge \((x,y)\) in forward direction*

*avg # times take edge \((x,y)\) in backward direction*

**Note on normalization**

What is \( i_{a} \)?

Know net \# times take some edge out of a

\[ = \sum_{y \neq a} i_{ay} = 1 \text{ (by def. of problem)} \]

Scale so this is true.

I.e. If \( \alpha = \# \text{units current flowing out of a} \)

when set \( v(a) = 1, v(b) = 0 \), then

\[ v_a = \frac{1}{\alpha} \]

(\( \text{so would then flow } \alpha \cdot \frac{1}{\alpha} = 1 \)).

\( \frac{1}{\alpha} \) is called the effective resistance between \( a \) and \( b \).