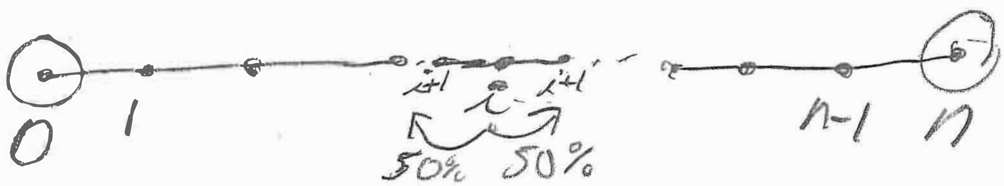
18.095 Lecture 1

See slides for beginning of lecture.

Warmup—Drunkard's walk (described in slides)



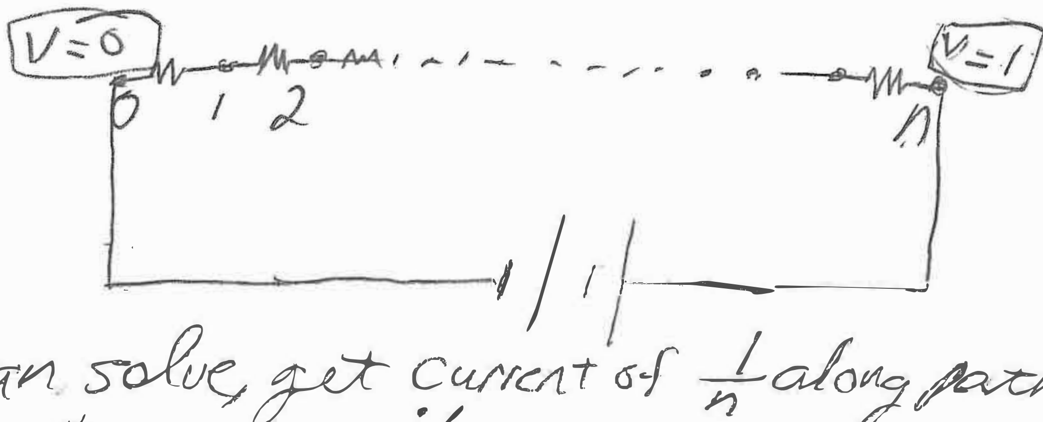
Let $p(i)$ = prob get to n before getting to 0 if start at i.

1. $p(0)=0$
2. $p(n)=1$
3. for $i=1,2,\cdots n-1$,

$$p(i)=\frac{1}{2}p(i+1)+\frac{1}{2}p(i-1) \quad \text{[why?]}$$

Can solve for $p(i)$, get $p(i)=i/n$ for $i=0,1,\cdots n$

Circuit—



Can solve, get current of $\frac{1}{n}$ along path, voltage of i^{th} vertex $= i/n$.

Same answer as above. Is this a coincidence?

Discrete Harmonic Functions

Have graph w/ 2 kinds of verts, boundary verts and internal verts. [Assume set of boundary verts $\neq \emptyset$]

A function f: verts $\to \mathbb{R}$ is a (discrete) harmonic function if, for every internal node x,

$$f(x) = \sum_{y \sim x} f(y)\, \underbrace{w_{xy}}_{\text{wt. of edge }(x,y)} / \deg(x)$$

"weighted average of its neighbors"

where $\deg(x) = \sum_{y \sim x} w_{xy}$

Maximum Principle - Harmonic fn. on a connected graph achieves its max (and min) on boundary.

Proof - Suppose otherwise. Let S = set of nodes achieving max

Note S consists entirely of internal nodes, by assumption.

$\bar{S} \neq \emptyset$ (since contains all bdry verts.)

$\exists$ an edge (x,y) w/ $x \in S$, $y \in \bar{S}$ (since graph is connected)

$f(x) > f(y)$, since x achieves max & y doesn't.

None of x's nbrs have bigger val than $f(x)$, since $f(x)$ is max.

$\Rightarrow f(x)$ is (weighted) avg of numbers all $\leq f(x)$, with at least one strictly less than $f(x)$, which can't happen. This is a contradiction to assumption f is harmonic. [Analogous argument works for min too]

Corollary [Uniqueness] — Suppose f, g harmonic functions and $f(x) = g(x)$ for all boundary verts x. Then $f(x) = g(x)$ for all x (i.e. for internal verts too).

Proof — Let $h(x) = f(x) - g(x)$. Can check h is harmonic. h is 0 on all boundary verts $\Rightarrow$ $\max(h) = \min(h) = 0$ by maximum principle.

Voltage is harmonic — Say fix $v(a) = 1$, $v(b) = 0$, & let current flow, use Kirchoff's laws to determine voltage @ all other verts.

Claim — v is harmonic (with boundary $= \{a, b\}$)

Why? — For edge (x, y), $i_{xy} = \dfrac{v(x) - v(y)}{r_{xy}} = (v(x) - v(y)) w_{xy}$

[Recall $w_{xy} := \dfrac{1}{r_{xy}}$]

By fact that current in = current out, for every internal vertex x, $0 = \sum_{y \sim x} i_{xy} = \sum_{y \sim x} (v(x) - v(y)) w_{xy}$

$$\Rightarrow v(x) \sum_{y \sim x} w_{xy} = \sum_{y \sim x} v(y) w_{xy}$$

$$\Rightarrow v(x) = \frac{\sum_{y \sim x} v(y) w_{xy}}{\sum_{y \sim x} w_{xy}}$$

$\leftarrow = \deg(x)$

Answer to walk question is harmonic

Let $p(x)$ = prob. that walk starting @ x reaches a before it reaches b.

If @ x, prob. go to $y = \dfrac{w_{xy}}{\deg(x)}$

Have $p(a)=1$

$p(b)=0$

$$p(x) = \sum_{y \sim x} \frac{w_{xy}}{\deg(x)} \cdot p(y)$$

prob if @ x go to y in next step

prob hit a before b if start @ y

So.....

p & v harmonic w/ same bdry vals

$\Longrightarrow p = v$.

I.e. - Prob. hit a before b when starting @ x

$=$ voltage if turn graph into circuit, tie battery to it st. voltage of $a=1$, voltage of $b=0$!

If multiple absorbing states & want to know prob get to states $1, \dots, t$ before states $t+1, \dots, k$ just set $v(x)=1$ for states $1, \dots, t$, and $v(x)=0$ for states $t+1, \dots, k$ 5

[Why?]

Know what voltage means. What about current?

Claim — i_{xy} = net freq. walk that starts @ a goes through edge (x,y) before getting to b. [Need to normalize correctly — see note @ end]

Why? — Let $u(x)$ = avg. # times walk starting @ a visits x before getting to b.

$u(b)=0$

For internal vertex x, $u(x) = \sum_y u(y) \underbrace{\left[\text{prob. go to } x \text{ if } @ y\right]}_{= w_{xy}/\deg(y)}$

u not harmonic, but will show $s(x) = u(x)/\deg(x)$ is:

$$s(x) = \sum_y \frac{u(y)\, w_{xy}}{\deg(x) \deg y} = \sum_y u(y) \frac{w_{xy}}{\deg(x)},$$ so

s harmonic.

Let $v_a = u(a)/\deg(a) = s(a)$ [not nec. = 1]

By uniqueness, s = voltages v when set $v(a)=v_a$, $v(b)=0$

So $i_{xy} = (v(x)-v(y))w_{xy} = (s(x)-s(y))w_{xy}$

$$= \left[\frac{u(x)}{deg(x)} - \frac{u(y)}{deg(y)}\right] w_{xy}$$

$$= \underbrace{u(x)\frac{w_{xy}}{deg(x)}}_{\substack{\text{avg \# times}\\ \text{take edge }(x,y)\\ \text{in forward direction}}} - \underbrace{u(y)\frac{w_{xy}}{deg(y)}}_{\substack{\text{avg \# times}\\ \text{take edge }(x,y)\\ \text{in backward direction.}}}$$

<u>Note on normalization-</u>

what is V_a?

know net # times take some edge out of a

$$= \sum_{y\sim a} i_{ay} = 1 \quad \text{(by def of problem)}$$

Scale so this is true.

I.e., If α = # units current flowing out of a when set $v(a)=1$, $v(b)=0$, then

$V_a = 1/\alpha$ (so would then flow $\alpha \cdot \frac{1}{\alpha} = 1$.

$\frac{1}{\alpha}$ is called <u>the effective resistance</u> between a and b.