Continued Fraction Notes

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Continued fractions are a topic in number theory which has applications to rational approximations of real numbers. We will first explain what a continued fraction is, prove some basic theorems about them, and then show how they can be used to find good rational approximations.

A continued fraction is a number of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}}$$

Note that the numerator of all these fractions is 1. There are generalizations that don't require this; these are called *generalized* continued fractions. We will limit these notes to regular continued fractions, where the numerator is 1.

How do we find a continued fraction expansion of a number? It is relatively straightforward. As an example, let's find the continued fraction expansion of $\frac{37}{112}$. We start by dividing 112 by 37 and get a remainder of 11. This gives

$$\frac{37}{112} = \cfrac{1}{3 + \cfrac{11}{37}}$$

We then just keep expanding the last piece of the denominator in the same way:

$$\frac{37}{112} = \cfrac{1}{3 + \cfrac{11}{37}} = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{4}{11}}} = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{3}{4}}}} = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3}}}}}$$

We will denote this as $\frac{37}{112} = CF[0, 3, 3, 2, 1, 3]$. Here 0 is the integer part of our number; $\frac{149}{112}$ would be $CF[1, 3, 3, 2, 1, 3]$.

The following theorem is not hard to prove, so we will leave it as an exercise for the reader:

Theorem 1. *A continued fraction expansion of R terminates if and only if R is a rational number.*

If a number is not rational, it has a unique infinite continued fraction.

Let's look at the continued fraction expansion of the golden ratio: $\frac{\sqrt{5}+1}{2}$. We have

$$\frac{\sqrt{5}+1}{2} \ = \ 1+\frac{\sqrt{5}-1}{2} \ = \ 1+\frac{(\sqrt{5}-1)(\sqrt{5}+1)}{2(\sqrt{5}+1)} \ = \ 1+\frac{1}{\dfrac{\sqrt{5}+1}{2}}$$

But we can now use this equation to replace the $\frac{\sqrt{5}+1}{2}$ in the denominator to get

$$\frac{\sqrt{5}+1}{2} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$$

or $\frac{\sqrt{5}+1}{2} = CF[1,1,1,1,1,\ldots]$.

This is a special case of the theorem

Theorem 2. *A continued fraction expansion of R is eventually periodic if and only if R is a solution to a quadratic equation with integer coefficients, i.e., $R = \frac{a+\sqrt{b}}{c}$.*

What *eventually periodic* means here is that after an initial segment, it starts repeating.

Before we prove this theorem, let's give an example for the forward direction that shows how the proof works. Let's look at the continued fraction $CF = [0,3,5,3,5,3,5,\ldots]$. That is,

$$\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{3+\cfrac{1}{5+\ldots}}}}$$

Let its value be x. Then, we have

$$x = \cfrac{1}{3+\cfrac{1}{5+x}}.$$

We can simplify this, We get

$$x = \frac{5+x}{16+3x},$$

or

$$x(16+3x) = 5+x.$$

This gives the quadratic equation $3x^2+15x-5=0$, which has solutions $\frac{-15\pm\sqrt{285}}{6}$. Since the solution is positive, it must be $\frac{\sqrt{285}-15}{6}$.

In fact, any expression of the form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \; \cfrac{1}{a_k + x}}}}},$$

where $a_1, a_2, \dots a_k$ are nonnegative integers, reduces to a fraction of the form $\frac{rx+s}{tx+u}$ where r, s, t, u are integers. Setting

$$x = \frac{rx+s}{tx+u}$$

gives a quadratic equation $tx^2 + (n-r)x - s = 0$, which has solutions of the form $\frac{a \pm \sqrt{b}}{c}$.

How about the other direction? Let's look at the continued fraction for $\frac{1+\sqrt{17}}{2}$. We have

$$\frac{\sqrt{17}+1}{2} = 2+\tfrac{1}{2}(\sqrt{17}-3) = 2+\frac{(\sqrt{17}-3)(\sqrt{17}+3)}{2(\sqrt{17}+3)} = 2+\frac{1}{\frac{1}{4}(\sqrt{17}+3)} = 2+\frac{1}{1+\frac{1}{4}(\sqrt{17}-1)}$$

Continuing with similar calculations (which we omit), we find

$$2 + \frac{1}{1+\frac{1}{4}(\sqrt{17}-1)} = 2 + \cfrac{1}{1 + \cfrac{1}{1+\frac{1}{4}(\sqrt{17}-3)}} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3+\frac{1}{2}(\sqrt{17}-3)}}},$$

and since we saw $\frac{1}{2}(\sqrt{17}-3)$ earlier in our calculations, we see that the continued fraction has begun to repeat. Thus, $\frac{1}{2}(\sqrt{17}+1) = CF(2,1,1,3,1,1,3,1,1,3,1,1,\dots)$.

It's clear from the calculations that the remainder on the ith step will be of the from $\frac{a+b\sqrt{17}}{c}$. What we have to show is that a, b, and c are integers that are bounded, i.e., they don't grow to infinity as the level of the continued fraction increases. If we can show that, then by the pigeonhole principle they must start repeating at some point, and we have an eventually periodic continued fraction.

How do we get around the fact that a, b, c, might keep growing indefinitely? The first proofs of this theorem were quite complicated, but I found a beautiful way to prove this in a set of lecture notes by Aaron Pollack. It depends on the following lemma:

Lemma 1. *Any fraction of the form*

$$\frac{b\sqrt{d}-a}{c},$$

where a, b, c, d are nonnegative integers can be written in the form

$$\frac{\sqrt{d'}-p}{q}$$

where p, q, d' are nonnegative integers and $q \mid d' - p^2$.

(Here the symbol $s \mid t$ means "s divides t".)

Proof. The proof of this is relatively simple. We let $q = c^2$, $d' = c^2b^2d$, and $p = ac$. Then

$$\frac{\sqrt{d'}-p}{q} = \frac{\sqrt{b^2c^2d}-(ac)^2}{c^2} = \frac{b\sqrt{d}-a}{c}$$

and $d' - p^2 = c^2b^2d - (ac)^2$, which is divisible by c^2. $\square$

Now, we are ready to prove the theorem.

Theorem 3. *Suppose we have a quadratic number of the form $\frac{\sqrt{d}-p}{q}$ where $q \mid d - p^2$. Then the continued fraction associated with it is eventually periodic.*

Proof. We will show that one step of the continued fraction algorithm starting with $\frac{\sqrt{d}-p}{q}$ with $q \mid d - p^2$ will take you to another continued fraction with $\frac{\sqrt{d}-p'}{q'}$ with $q' \mid d - p'^2$, keeping the same d. We also show that for any d, there are only a finite number of p and q satisfying $q \mid d - p^2$, meaning that by the pigeonhole principle, at some point we must come to a triple (d, p, q) that we have seen before. At this point, the continued fraction starts repeating.

First, why are there only a finite number of (p, q) with $q|d - p^2$? It's clear that $p < \sqrt{d}$ and $q < d$, so there are only a finite number of possible such p and q.

Next, we need to show that if we start with such a $\frac{\sqrt{d}-p}{q}$, we get on the next step a $\frac{\sqrt{d}-p'}{q'}$. For a continued fraction step, the first thing we do is take the reciprocal:

$$\left(\frac{\sqrt{d}-p}{q}\right)^{-1} = \frac{q}{\sqrt{d}-p} = \frac{q(\sqrt{d}+p)}{d-p^2}$$

Now since we have $q \mid d - p^2$, we can let $q' = \frac{d-p^2}{q}$, and we have

$$\left(\frac{\sqrt{d}-p}{q}\right)^{-1} = \frac{\sqrt{d}+p}{q'},$$

and it is still true that $q' \mid d - p^2$.

The next step is to subtract an integer from this quantity to make it between 0 and 1. Let this integer be ℓ. this means we now have

$$\frac{\sqrt{d}+p}{q'} - \ell = \frac{\sqrt{d}+p-\ell q'}{q}.$$

Letting $p' = p - \ell q'$, it is still the case that $q' \mid d - p'^2$, because

$$d - p'^2 = d - p^2 + 2p\ell q' + \ell^2 q'^2,$$

which is a multiple of q' because $d - p^2$ is a multiple of q'. So each step of the continued fraction preserves the relation

$$q_i \mid d - p_i^2,$$

and by the pigeonhole principle, the continued fraction must be eventually periodic. $\square$

The last thing to do is show that if we have a continued fraction that cannot be put into the form $\frac{\sqrt{d}-p}{q}$ with $d > p^2$ and $q \mid d - p^2$, then after some number of continued fraction iterations, it will be in this form. I will leave this as an exercise.

0.1 Using Continued Fractions for Diophantine Approximation

Continued approximations can be used for finding good approximations of reals by fractions. For example, the approximations

$$\pi \approx 3,\ \pi \approx 22/7 \approx 3.143,\ \pi \approx \frac{333}{106} \approx 3.14151,$$

can all be found by looking at the *convergents* of the continued fraction for π.

The convergents of a continued fraction are the continued fractions made from an initial sequence of the elements. For example, we have

$$\frac{615}{2048} = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{33 + \cfrac{1}{1 + \cfrac{1}{5}}}}},$$

so $\frac{615}{2048} = CF(0, 3, 3, 33, 1, 5)$.

The convergents of this continued fraction are $CF(0)$, $CF(0, 3)$, $CF(0, 3, 3)$, $CF(0, 3, 3, 33)$, and $CF(0, 3, 3, 33, 1)$. Here are their numerical values:

$$\frac{1}{3} = \frac{1}{3} \approx\ 0.3333,$$

$$\cfrac{1}{3 + \cfrac{1}{3}} = \frac{3}{10} \approx\ 0.3,$$

$$\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{33}}} = \frac{100}{333} \approx 0.3003003,$$

$$\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{33+\cfrac{1}{1}}}} = \frac{103}{343} \approx 0.3002915,$$

You can see that these values keep getting closer to $615/2048 \approx .30029297$.

We now show that all the closest approximations to a real number are convergents of its continued fraction. Specifically, we show:

Theorem 4. *if* $|R - \frac{p}{q}| \leq \frac{1}{2q^2}$, *then* $\frac{p}{q}$ *is one of the convergents of* r.

How do we show this theorem? We first prove a lemma:

Lemma 2. *Suppose* $\frac{p}{q} < R < \frac{p'}{q'}$ *and* $pq' = 1 + p'q$. *With these conditions if* $q < q'$, *then* $\frac{p}{q}$ *is one of the convergents of* R, *and if* $q' < q$, *then* $\frac{p'}{q'}$ *is one of the convergents of* R.

Let's take as an example $R = 615/2048 \approx .30030293$. We have

$$\frac{3}{10} = 0.3 < 615/2048 = 0.30030293 < \frac{10}{33} = 0.30303.$$

We can easily check that $3 \cdot 33 + 1 = 10 \cdot 10$ This shows that $\frac{3}{10}$ is a convergent. ($\frac{33}{100}$ is not, although it is something called a *semiconvergent*).

Proof of Lemma 2:

First, let's look at the continued fractions for $\frac{p}{q}$ and $\frac{p'}{q'}$. I claim that they cannot be of the forms

$$\frac{p}{q} = \cfrac{1}{a+\cfrac{1}{b+\dots}}$$

and

$$\frac{p'}{q'} = \cfrac{1}{a'+\cfrac{1}{b'+\dots}}$$

with $a > a'$. Suppose they were. Then the fraction $\frac{1}{a}$ would be between $\frac{p}{q}$ and $\frac{p'}{q'}$, and would have a lower denominator than either, and it would be impossible for $\frac{p'}{q'} - \frac{p}{q} = \frac{1}{qq'}$. Thus, p' and q' both must start with $a = a'$. Now, let's consider the continued fractions

$$\frac{p}{q} = \frac{1}{a + \frac{b}{c}} \quad \text{and} \quad \frac{p'}{q'} = \frac{1}{a + \frac{b'}{c'}}$$

We will show $p'q - pq' = 1$ if and only if $b'c - bc' = 1$.

First, we calculate

$$\frac{p}{q} = \frac{c}{ac+b} \quad \text{and} \quad \frac{p'}{q'} = \frac{c'}{ac'+b'}$$

so

$$p'q - pq' = c(ac'+b') - c'(ac+b) = b'c - bc'.$$

What this shows is that if the two continued fractions

$$\frac{p}{q} = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \dots}}} \quad \text{and} \quad \frac{p'}{q'} = \cfrac{1}{a' + \cfrac{1}{b' + \cfrac{1}{c' + \dots}}},$$

satisfy $|\frac{p'}{q'} - \frac{p}{q}| = \frac{1}{qq'}$, then the continued fractions

$$\frac{r}{s} = \cfrac{1}{b + \cfrac{1}{c + \dots}} \quad \text{and} \quad \frac{r'}{s'} = \cfrac{1}{b' + \cfrac{1}{c' + \dots}},$$

satisfy $|\frac{r}{s} - \frac{r'}{s'}| = \frac{1}{ss'}$. We can in this way keep removing the first terms of the continued fractions and preserve the relation between the remaining terms. When can this process end? It can only end when one of the two continued fractions has been reduced to the form $\frac{1}{a}$. At this point, the other continued fraction must look like

$$\cfrac{1}{a + \cfrac{1}{b + \dots}},$$

so the first continued fraction is a convergent of the second one. And since R is sandwiched between them, the first continued fraction must also be a convergent of R.

We now use the lemma to prove the theorem. Suppose that $\frac{p}{q} < R$ (the case of $\frac{p'}{q'} > R$ is completely analogous) and that $R - \frac{p}{q} < \frac{1}{2q^2}$. Now, $\frac{p}{q}$ must be the closest fraction to R with denominator at most q, because two fractions with denominator less than or equal to q cannot be closer to each other than $\frac{1}{q(q-1)}$. There must also be a smallest fraction larger than $\frac{p}{q}$ with denominator at most q. Call this fraction $\frac{p'}{q'}$. Because there are no fractions between $\frac{p}{q}$ and $\frac{p'}{q'}$ with denominator at most q, we must have $pq' + 1 = p'q$. And we must have

$$\frac{p}{q} < R < \frac{p'}{q'}$$

Let's consider the fraction $\frac{p+p'}{q+q'}$. We have

$$\begin{aligned}\frac{p+p'}{q+q'} - \frac{p}{q} &= \frac{q(p+p') - p(q+q')}{q(q+q')} \\ &= \frac{p'q - q'p}{q(q+q')} \\ &= \frac{1}{q(q+q')} > \frac{1}{2q^2}\end{aligned}$$

This is larger than the distance between $\frac{p}{q}$ and R, so R must be between $\frac{p}{q}$ and $\frac{p+p'}{q+q'}$. And clearly the denominator q is less than the denominator $q+q'$. This shows that $\frac{p}{q}$ satisfies the conditions of Lemma 2 to be a convergent of R, and we have proved Theorem 4.