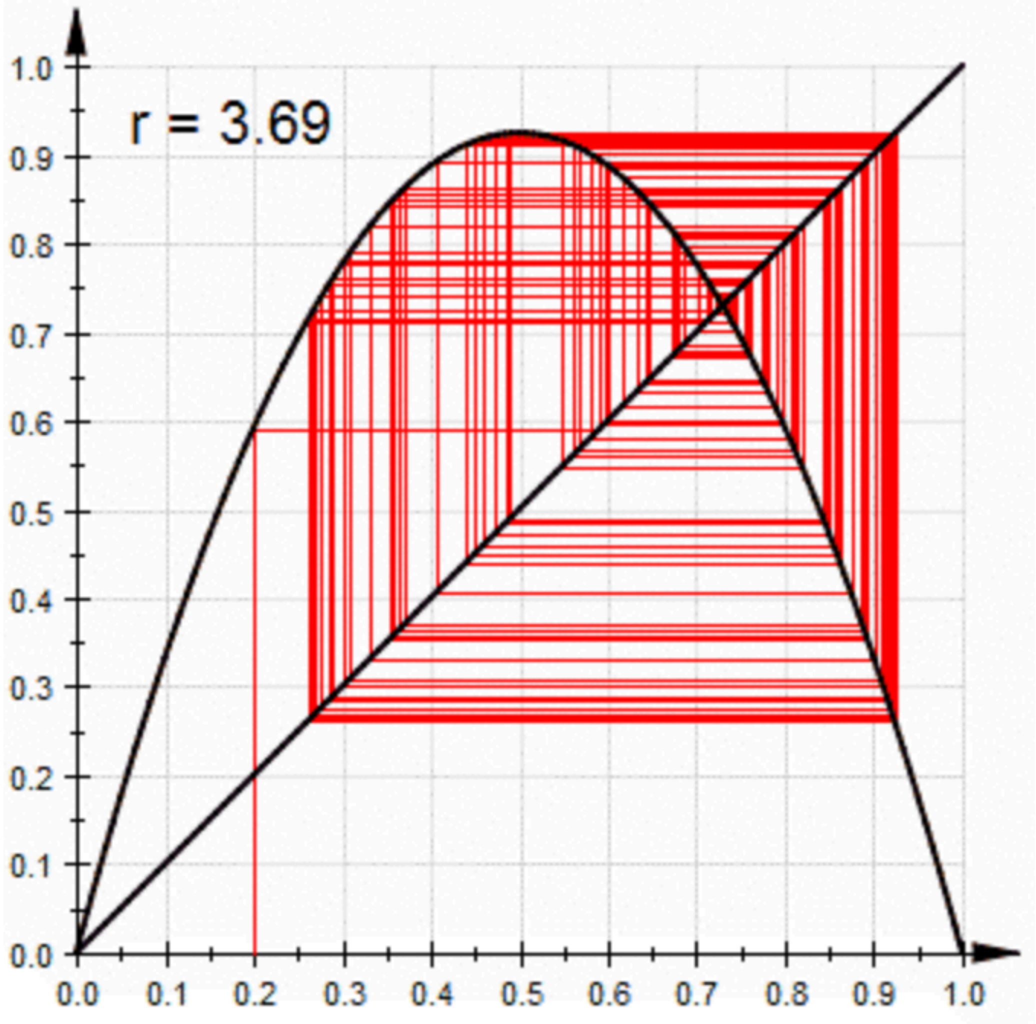
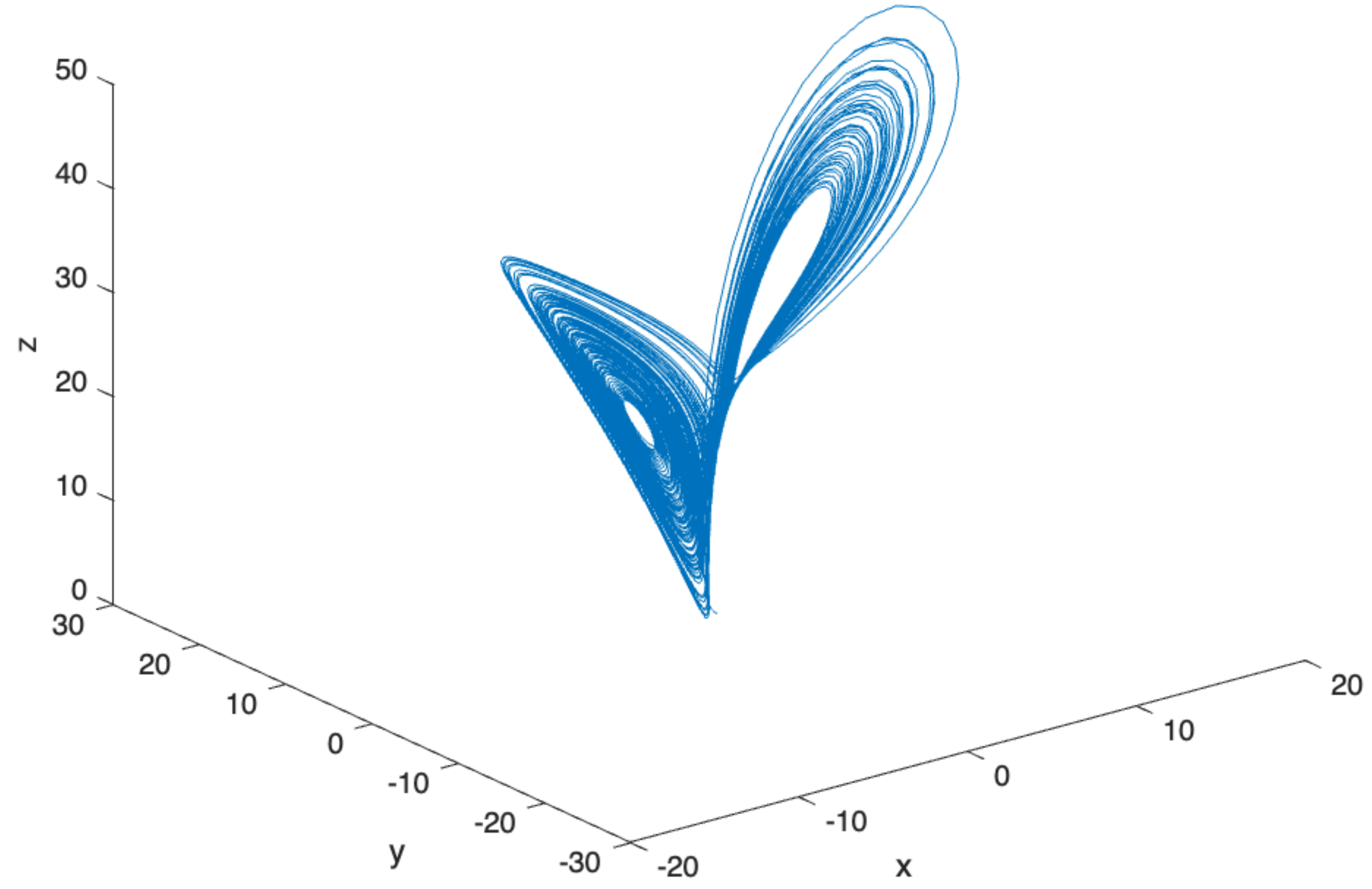
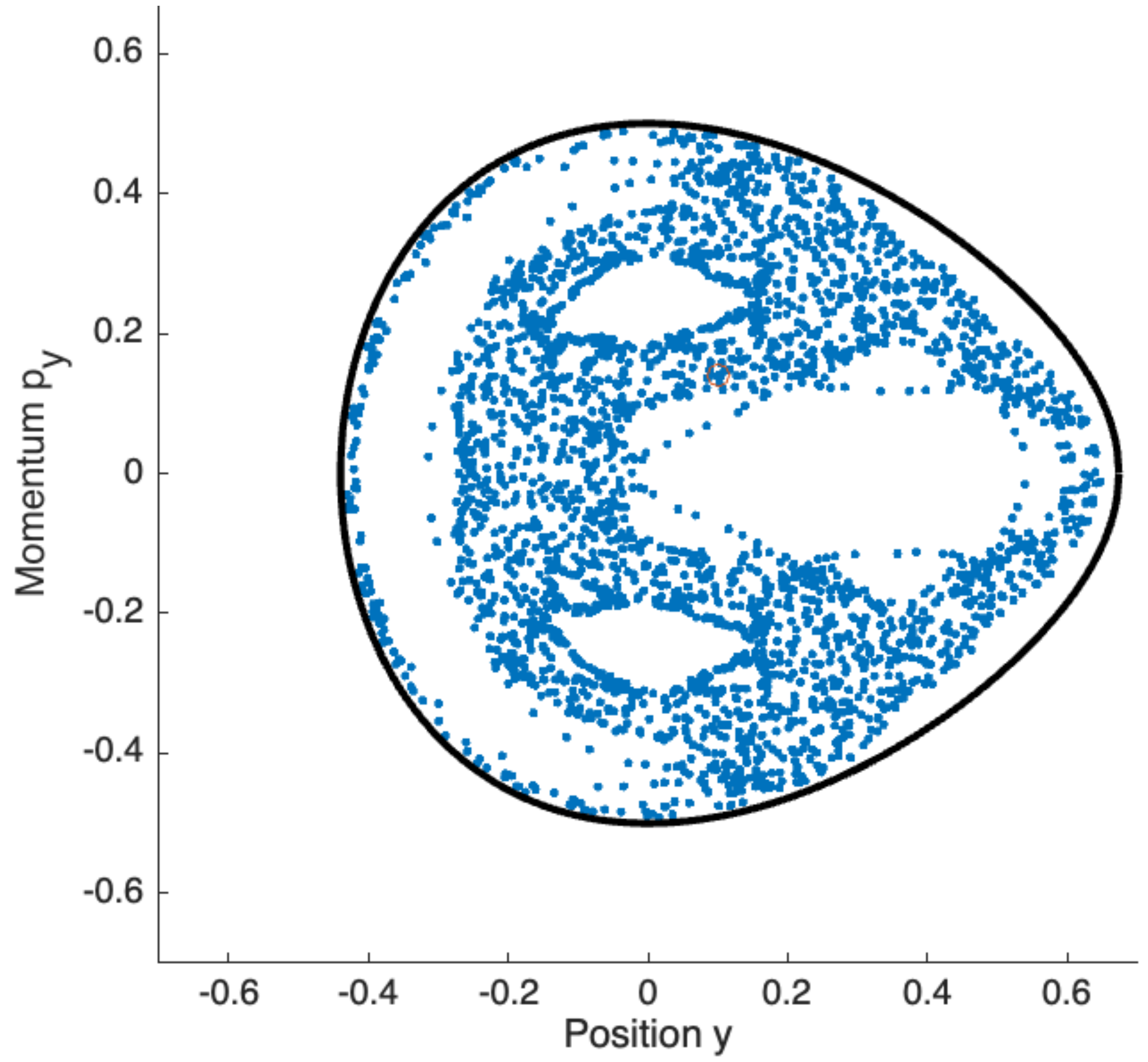
MIT
MATHEMATICS

IAP Lecture series 2024

Chaos in
discrete and continuous dynamical systems







Jörn Dunkel

Chaos

"Impossibility to make accurate long term predictions"

Discrete dynamical systems

$$x_{n+1} = f(x_n)$$

$$x_n = f \circ f \circ \cdots \circ f(x_0) = f^n(x_0)$$

Flow map / velocity field

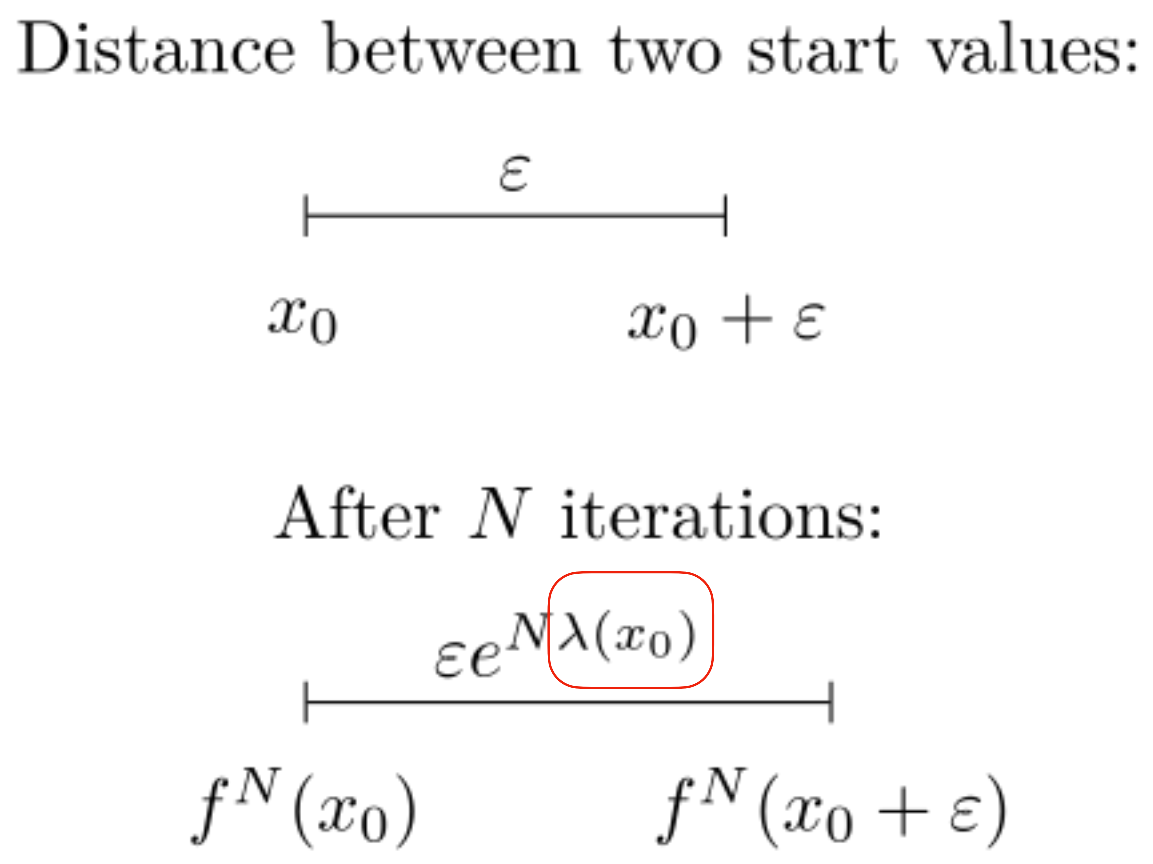
$$f : \mathbb{R}^d \to \mathbb{R}^d$$

Continuous dynamics

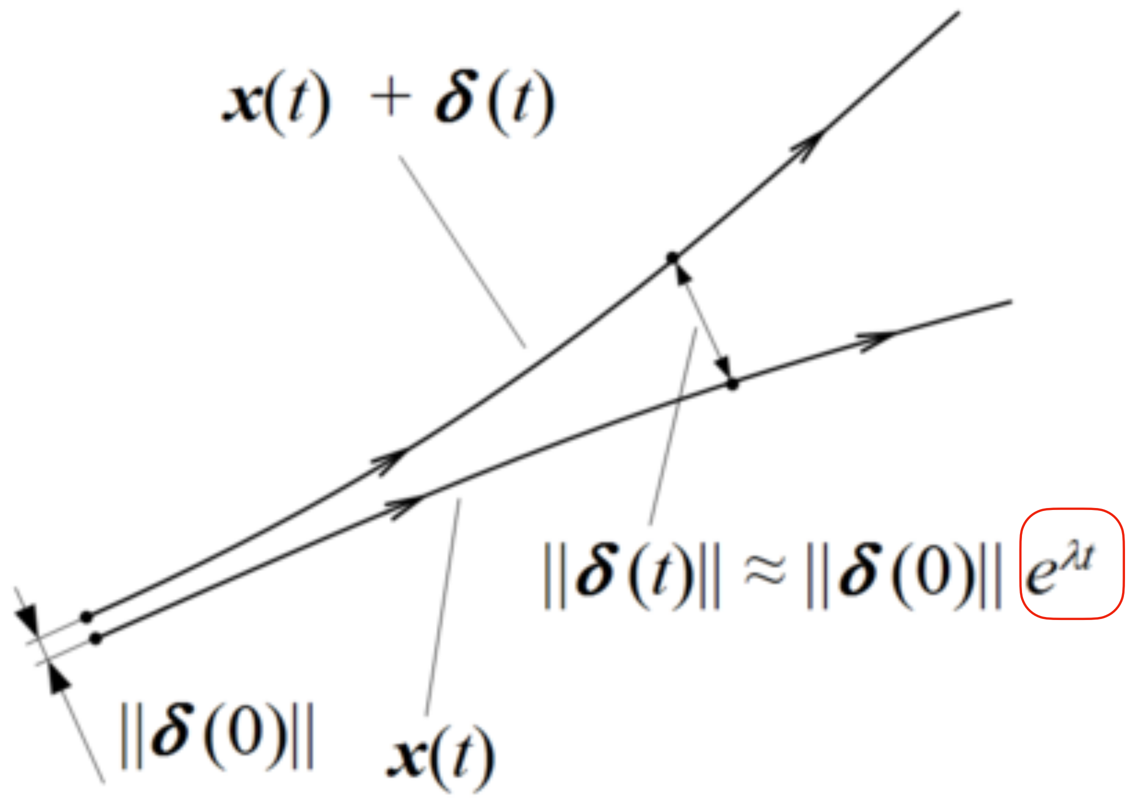
$$\dot{X}(t) = f(X(t))$$

$$X(0) = X_0$$

System chaotic iff distance between solutions diverges rapidly even when initial conditions very close



Maximal Lyapunov exponents



https://en.wikipedia.org/wiki/Lyapunov_exponent

I. Logistic map

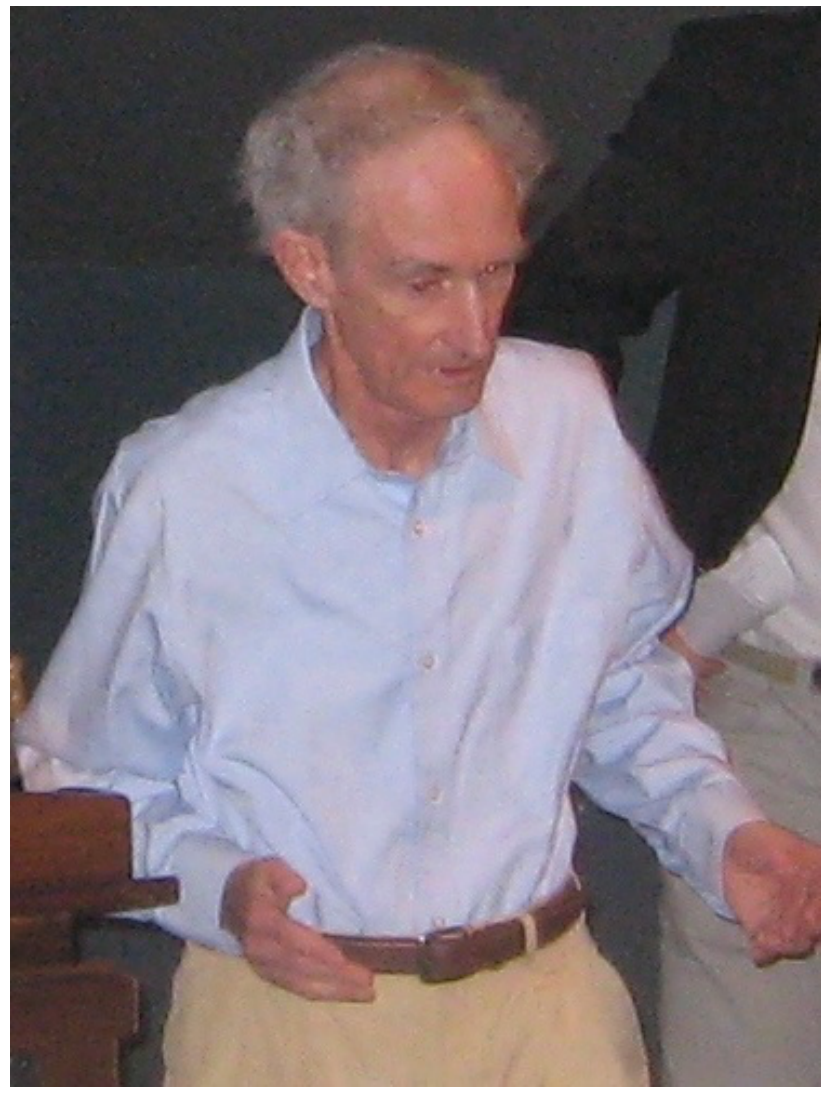
Discrete dynamics

Basic exponential growth model

$$x_{n+1} - x_n = g x_n$$

$$x_{n+1} = (1+g)x_n \quad = r x_n$$

$$\Rightarrow \quad x_n = (1+g)^n x_0 \quad = r^n x_n$$



(Slightly) more realistic model including competition for resources

Logistic map $$x_{n+1} = r(x_n - x_n^2) = r x_n(1 - x_n)$$

May, Robert M. (1976). "Simple mathematical models with very complicated dynamics". *Nature*. **261** (5560): 459–467

Fixed points

General definition $x_n = f(x_n)$

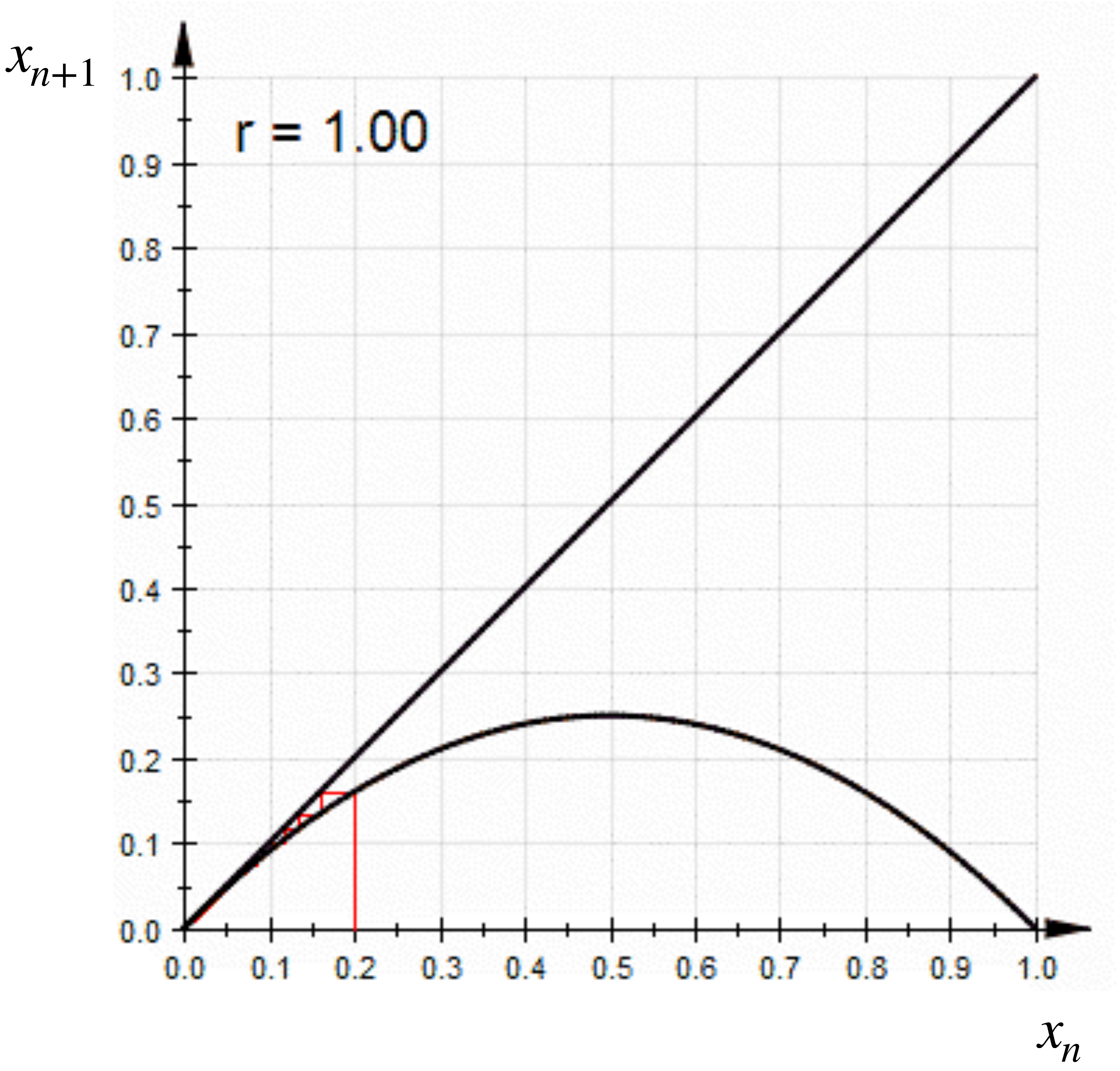
For logistic map $x_n = rx_n(1 - x_n)$

1. FP $x_- = 0$ **"Extinction"**

2. FP $x_n \neq 0$ $1 = r(1 - x_n)$ **Makes sense for r>1**

$$x_+ = 1 - \frac{1}{r} = \frac{r-1}{r}$$

Cobweb plots



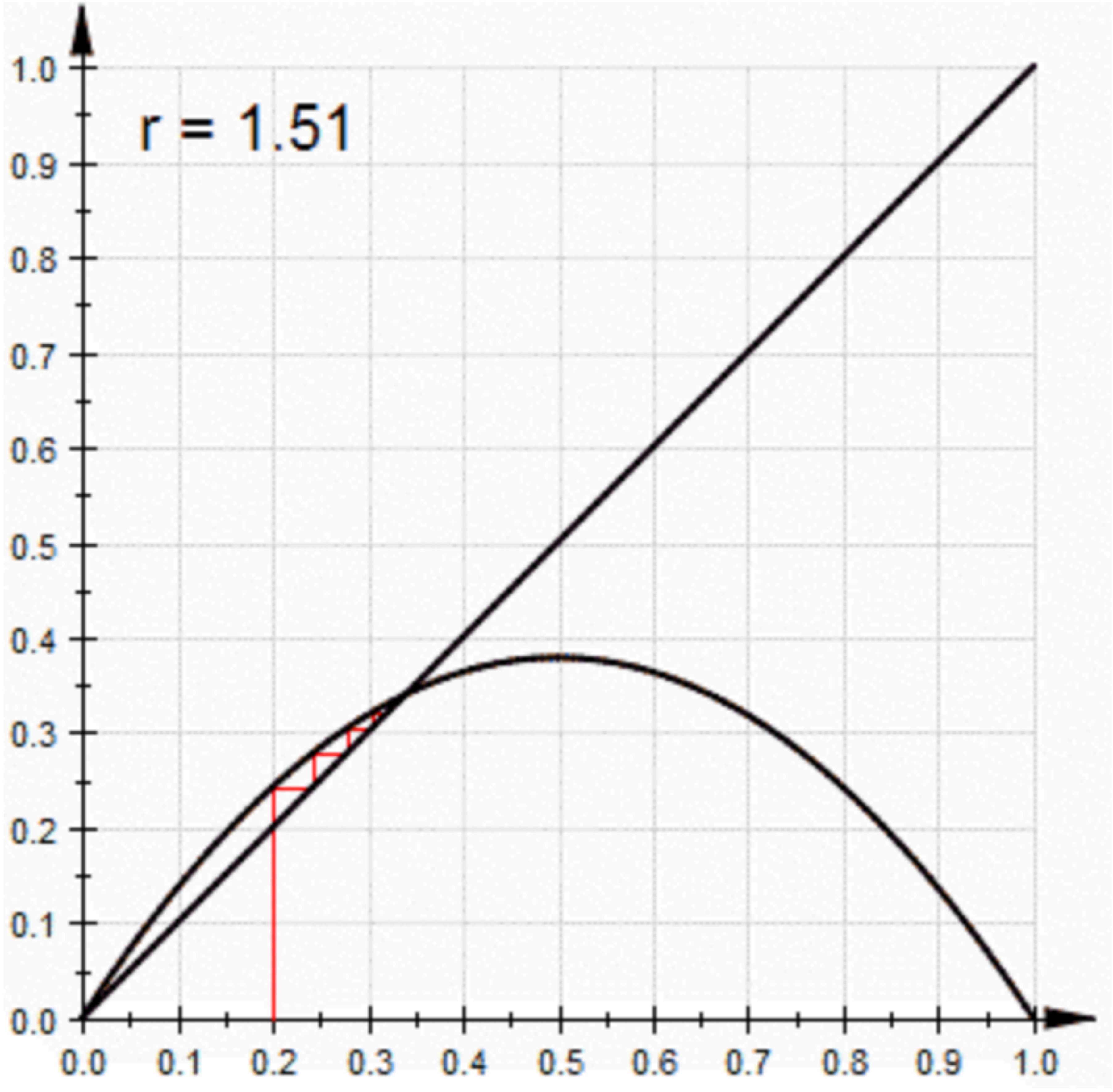
$0 \leq r \leq 1$

$x_{\_} = 0$
only FP
and stable

https://en.wikipedia.org/wiki/File:LogisticCobwebChaos.gif

Cobweb plots

x_{n+1}



x_n

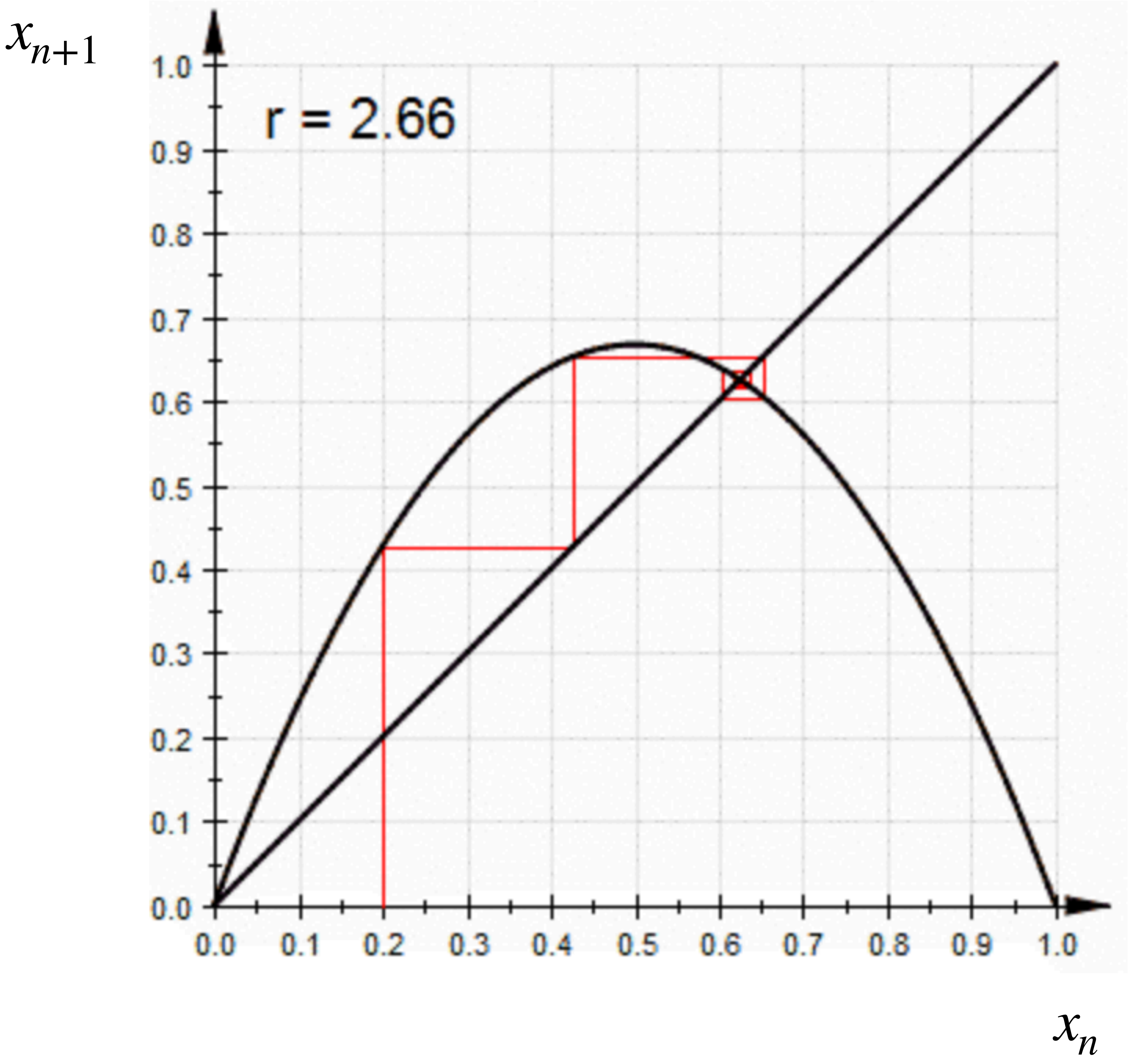
$$1 \leq r \leq 2$$

$$x_+ = \frac{r-1}{r}$$

stable FP
non-oscillatory approach

https://en.wikipedia.org/wiki/File:LogisticCobwebChaos.gif

Cobweb plots



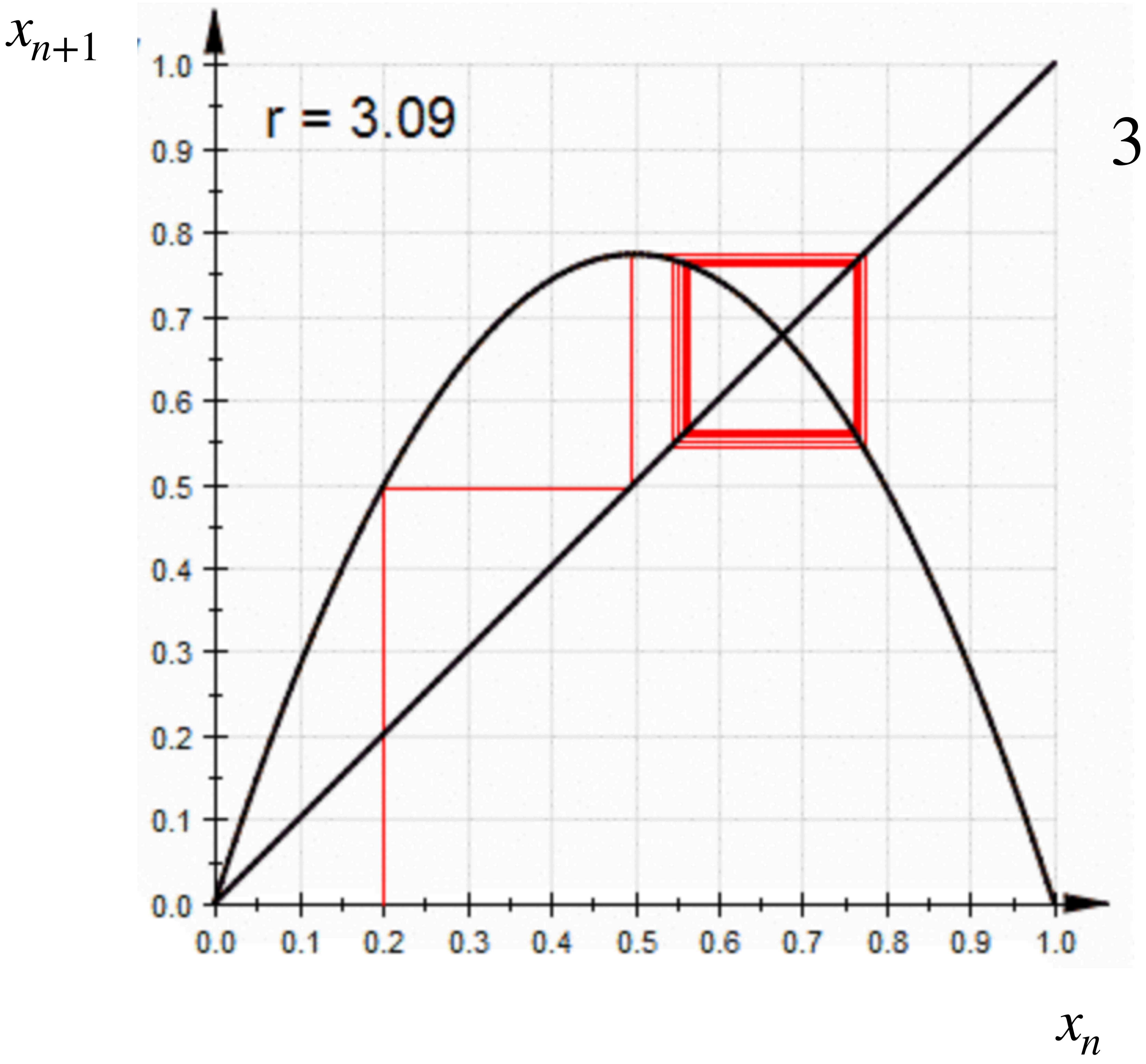
$$2 \leq r \leq 3$$

$$x_+ = \frac{r-1}{r}$$

stable FP
oscillatory approach

https://en.wikipedia.org/wiki/File:LogisticCobwebChaos.gif

Cobweb plots

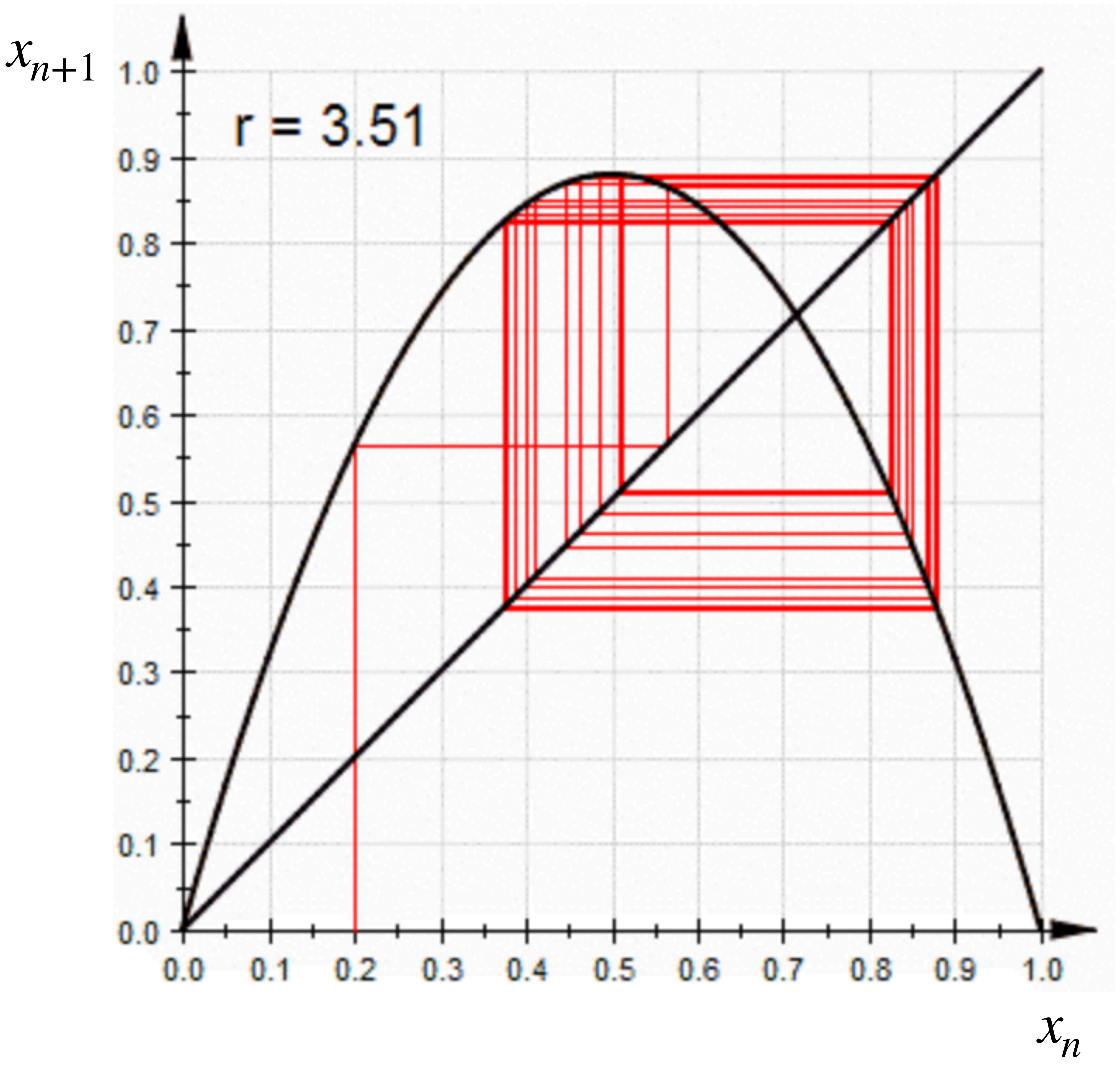


$$3 \leq r < 1 + \sqrt{6} \approx 3.44949$$

Steady-state oscillations between two values

https://en.wikipedia.org/wiki/File:LogisticCobwebChaos.gif

Cobweb plots

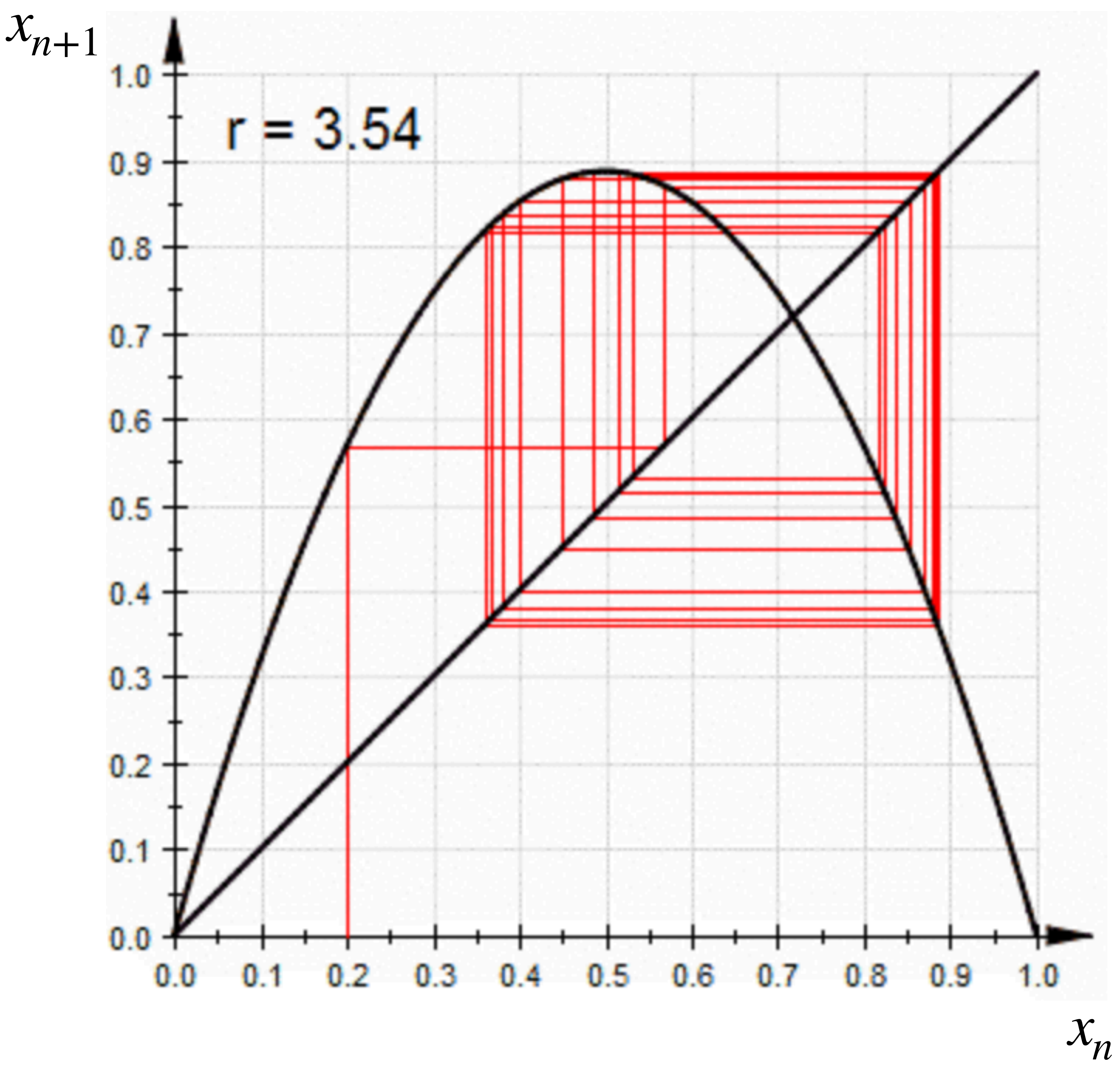


$3.44949 \lesssim r \lesssim 3.54409$

"Period doubling"
Steady-state oscillations
Among four values

https://en.wikipedia.org/wiki/File:LogisticCobwebChaos.gif

Cobweb plots



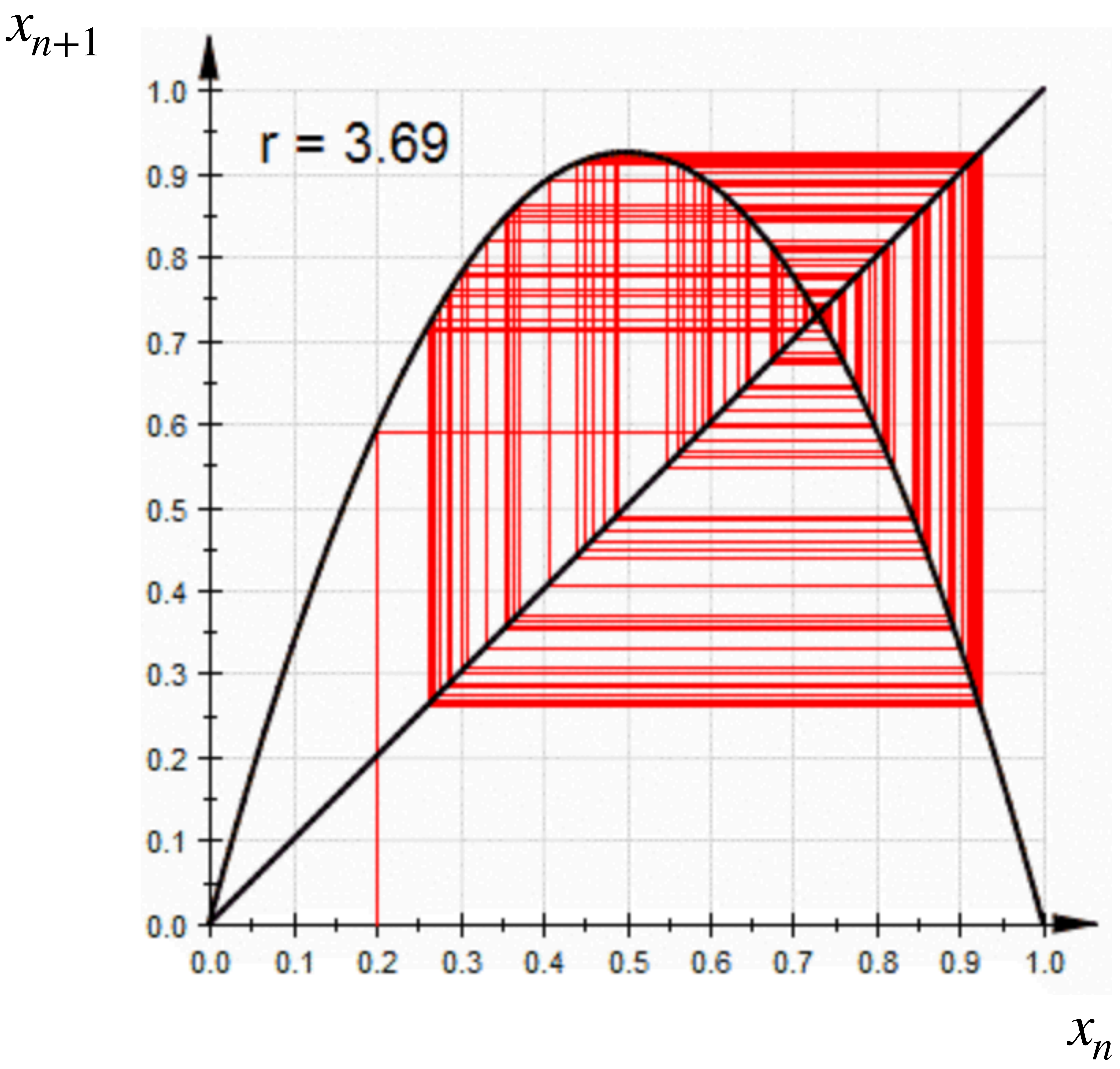
$$3.54409 \lesssim r \lesssim 3.56995$$

"Period doubling cascade"

Steady-state oscillations among 8, 16, … values

https://en.wikipedia.org/wiki/File:LogisticCobwebChaos.gif

Cobweb plots

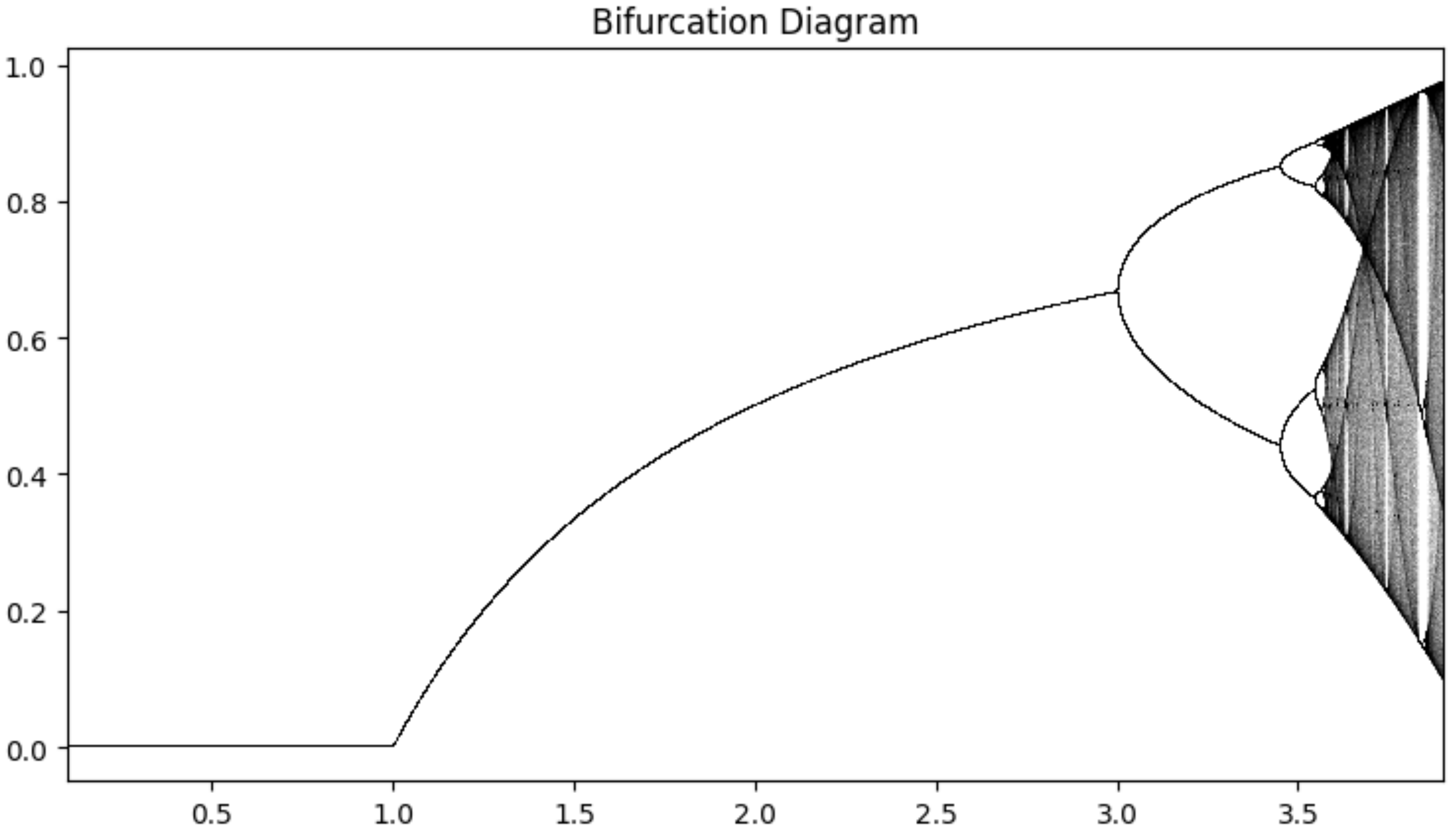


$r \gtrsim 3.56995$

"Chaos"

https://en.wikipedia.org/wiki/File:LogisticCobwebChaos.gif

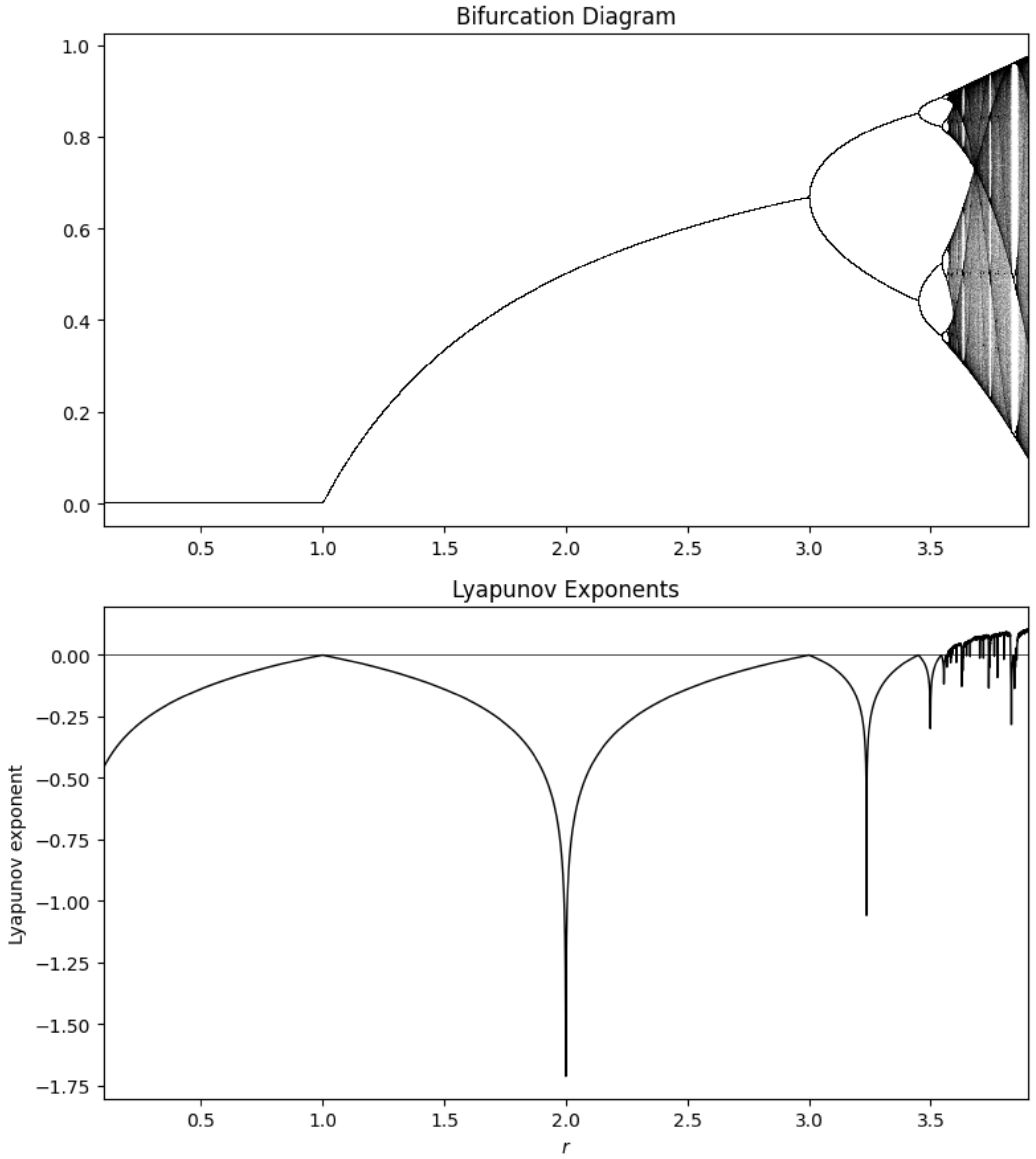
Feigenbaum plot



Homework 1: write a computer program that creates this plot

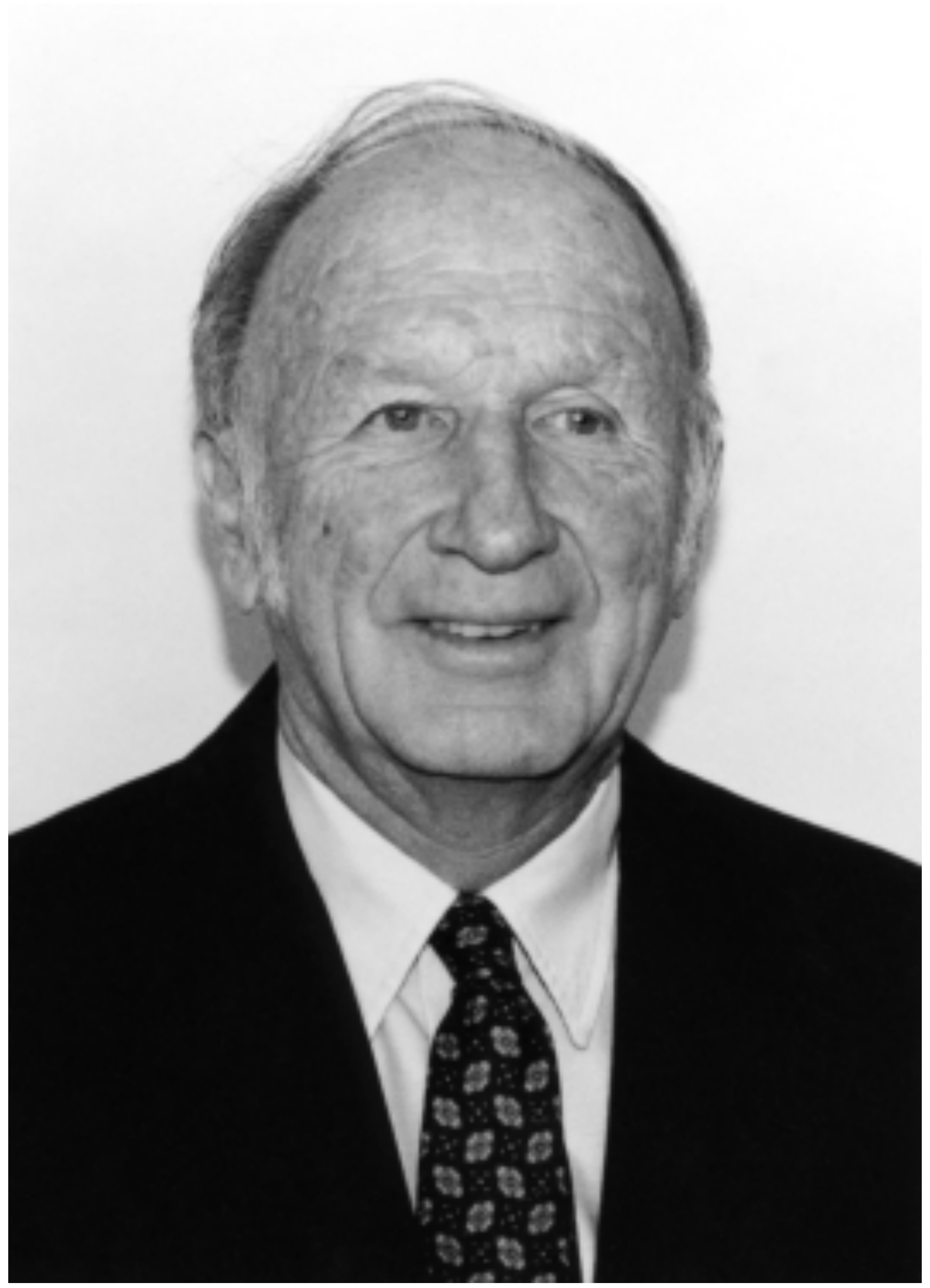
Lyapunov exponents

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$



II. Lorenz system

Lorenz system: dynamics



Edward Norton Lorenz

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z$$

https://en.wikipedia.org/wiki/Edward_Norton_Lorenz

https://en.wikipedia.org/wiki/Lorenz_system

Fixed points & linear stability

$$0 = \sigma(y - x)$$

$$0 = x(\rho - z) - y$$

$$0 = xy - \beta z$$

$$y = x$$

$$0 = x(\rho - z - 1)$$

$$0 = x^2 - \beta z$$

1. FP

$$x = y = z = 0$$

2. FP

$$z = \rho - 1$$

$$x = y = \pm\sqrt{\beta(\rho - 1)}$$

Homework 2: Evaluate the linear stability of these fixed points

Hodge-Helmholtz decomposition

$$\dot{x} = f_1 = \sigma(y - x)$$

$$\dot{y} = f_2 = x(\rho - z) - y$$

$$\dot{z} = f_3 = xy - \beta z$$

$$\mathbf{f} = \mathbf{G} + \mathbf{R}$$

Gradient field

$$\mathbf{G} = -\begin{pmatrix} \sigma x \\ y \\ \beta z \end{pmatrix}$$

$$\mathbf{G} = -\nabla\Phi$$

$$\Phi = \frac{1}{2}\left(\sigma x^2 + y^2 + \beta z^2\right)$$

Rotation fields

$$\mathbf{R} = \begin{pmatrix} \sigma x \\ x(\rho - z) \\ xy \end{pmatrix}$$

$$\nabla \cdot \mathbf{R} = 0$$

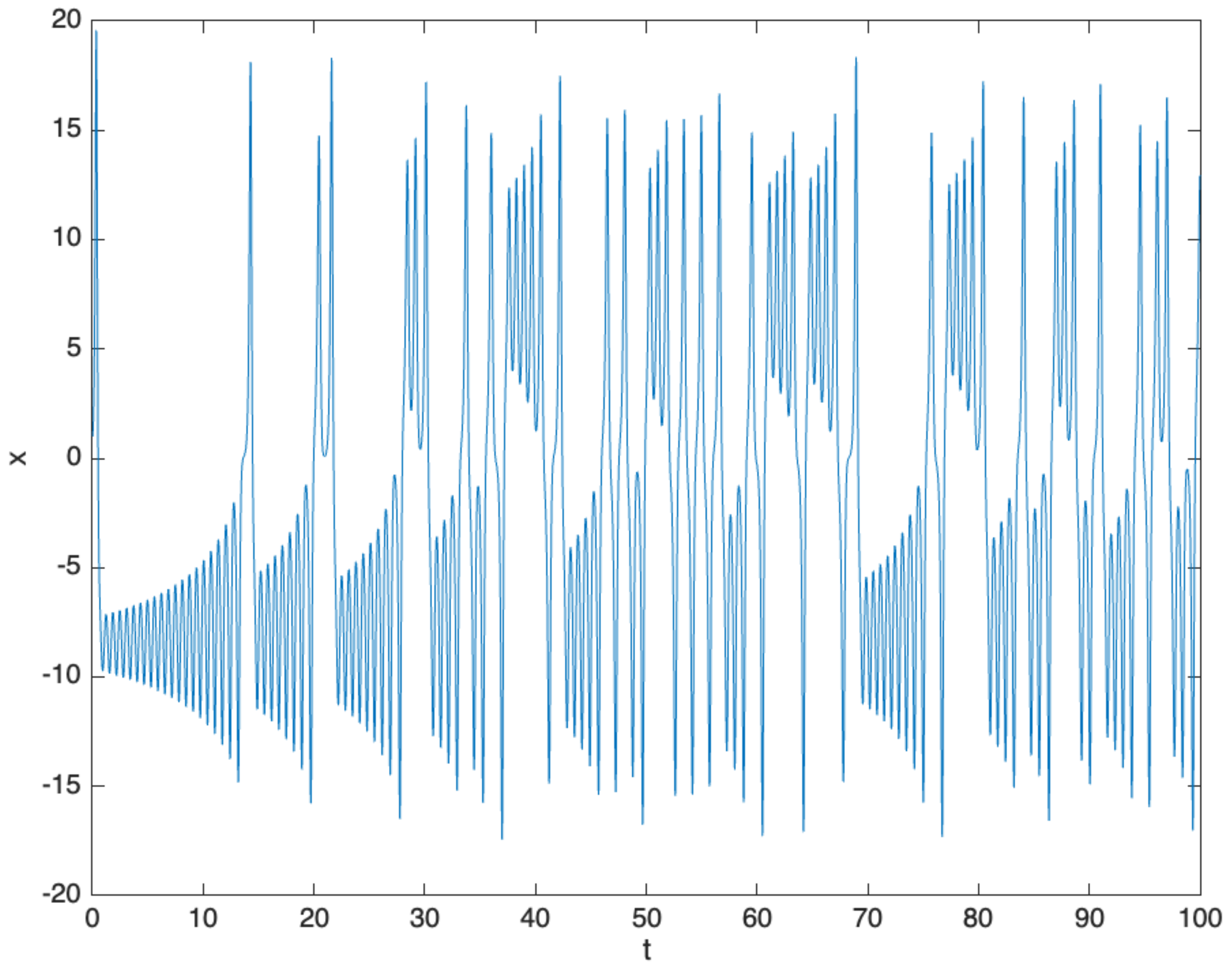
MATLAB code

```
% Solve over time interval [0,100] with initial conditions [1,1,1]

% ''f'' is set of differential equations
% ''a'' is array containing x, y, and z variables
% ''t'' is time variable

sigma = 10;
beta = 8/3;
rho = 28;

f = @(t,a) [-sigma*a(1) + sigma*a(2); rho*a(1) - a(2) - a(1)*a(3); -beta*a(3) + a(1)*a(2)];
[t,a] = ode45(f,[0 100],[1 1 1]);     % Runge-Kutta 4th/5th order ODE solver
```

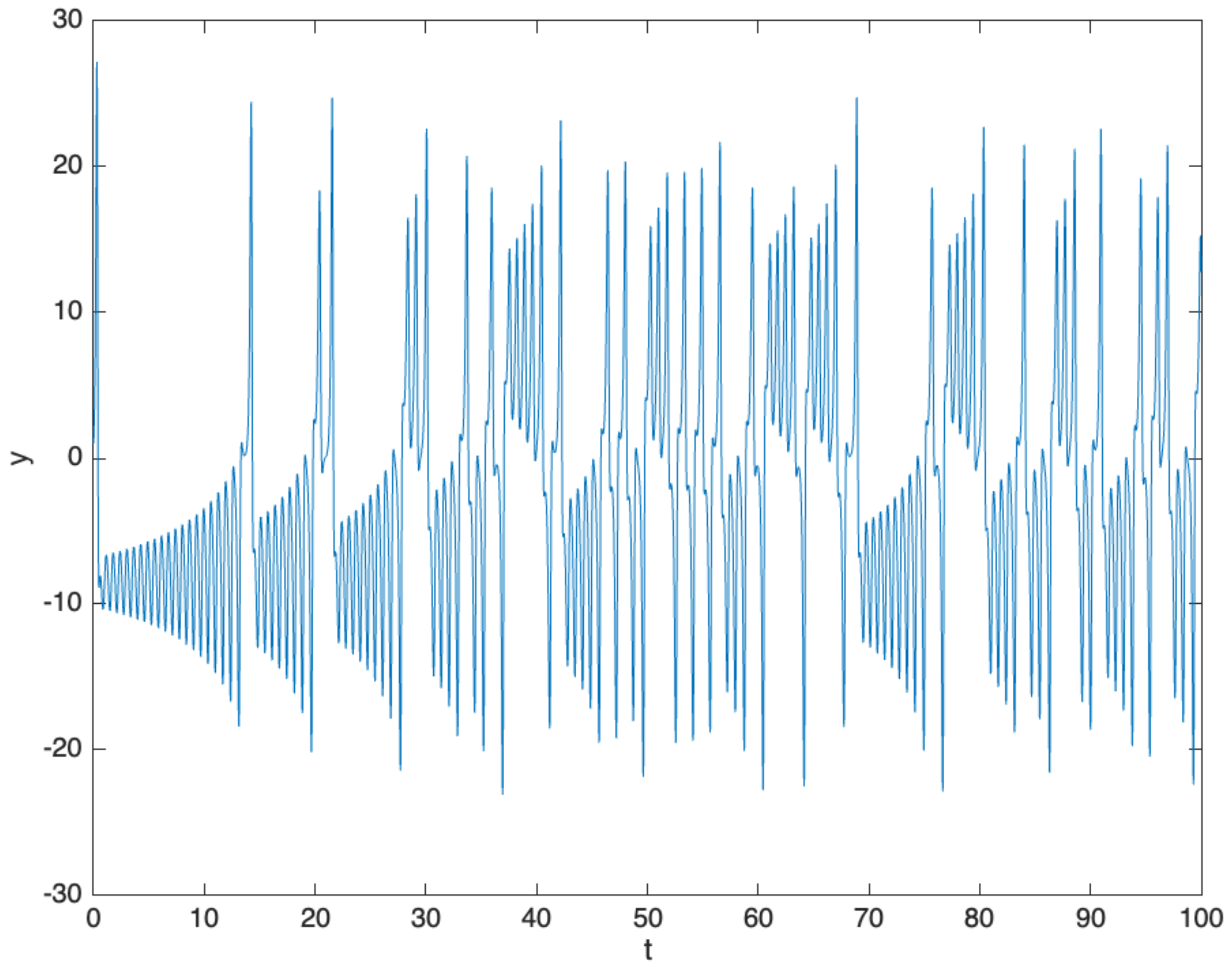
https://en.wikipedia.org/wiki/Lorenz_system

Simulation

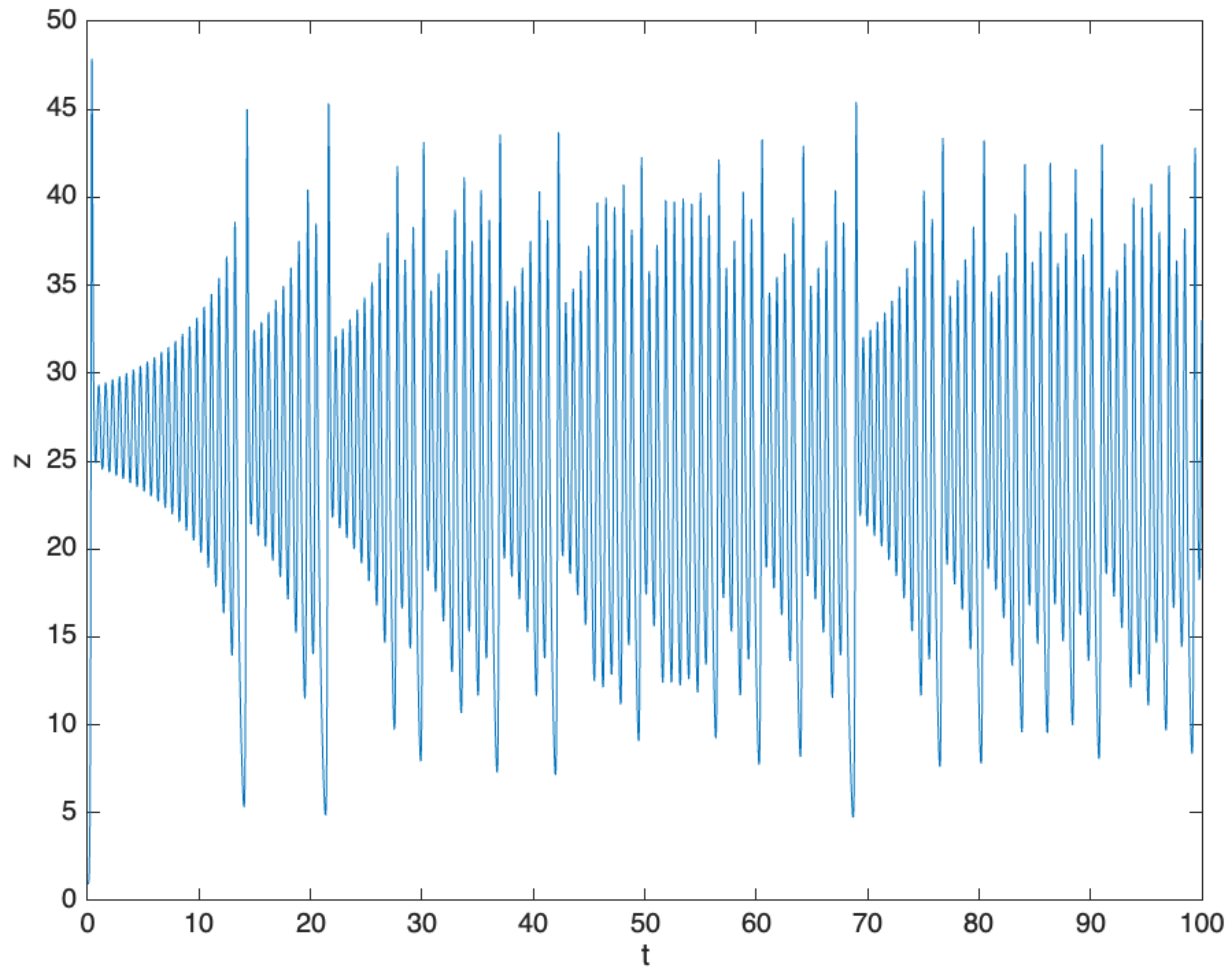
```
plot(t,a(:,1))
xlabel('t')
ylabel('x')
```




```
plot(t,a(:,2))
xlabel('t')
ylabel('y')
```

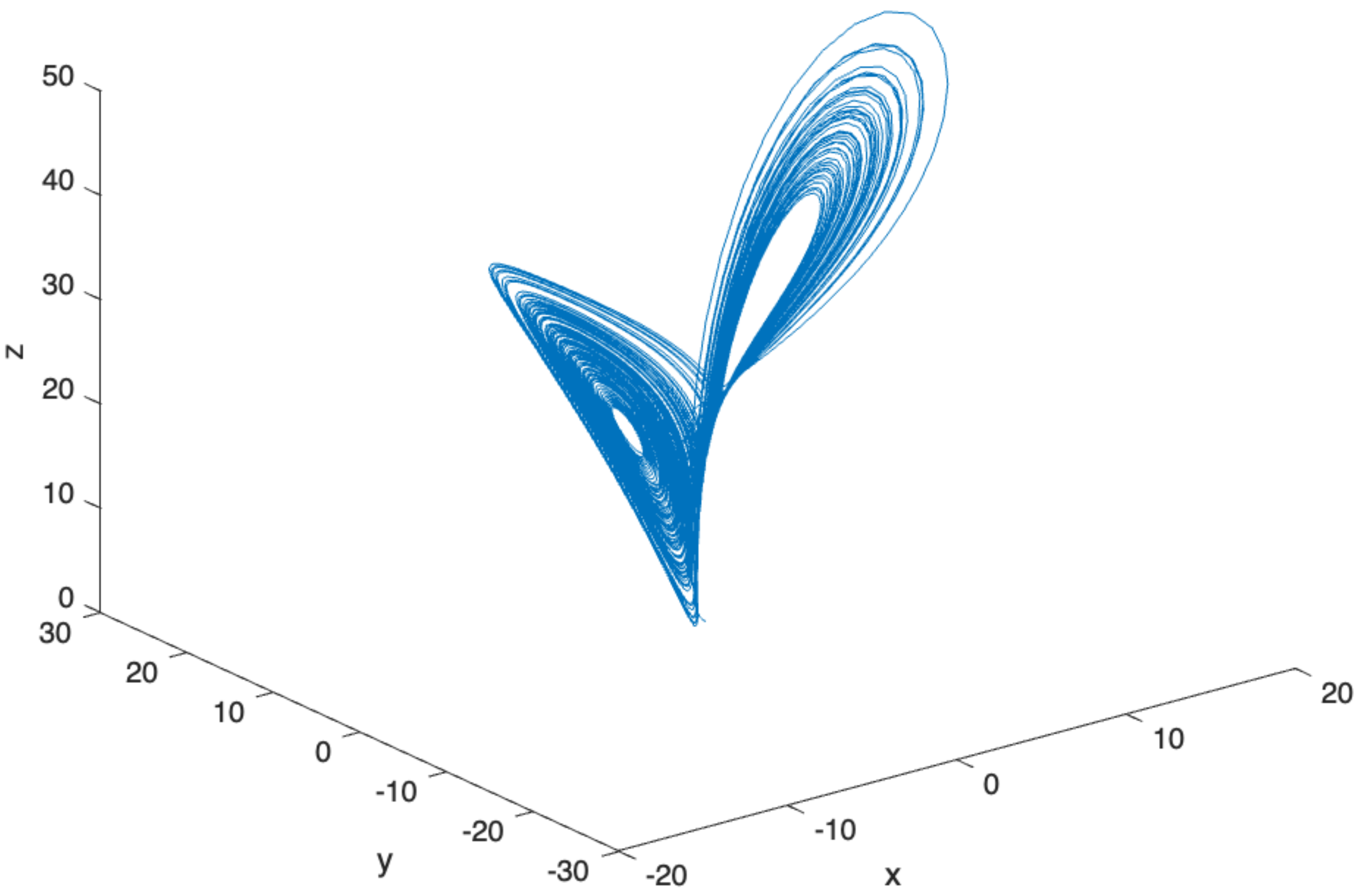



```
plot(t,a(:,3))
xlabel('t')
ylabel('z')
```



Simulation

```
plot3(a(:,1),a(:,2),a(:,3))
xlabel('x')
ylabel('y')
zlabel('z')
```



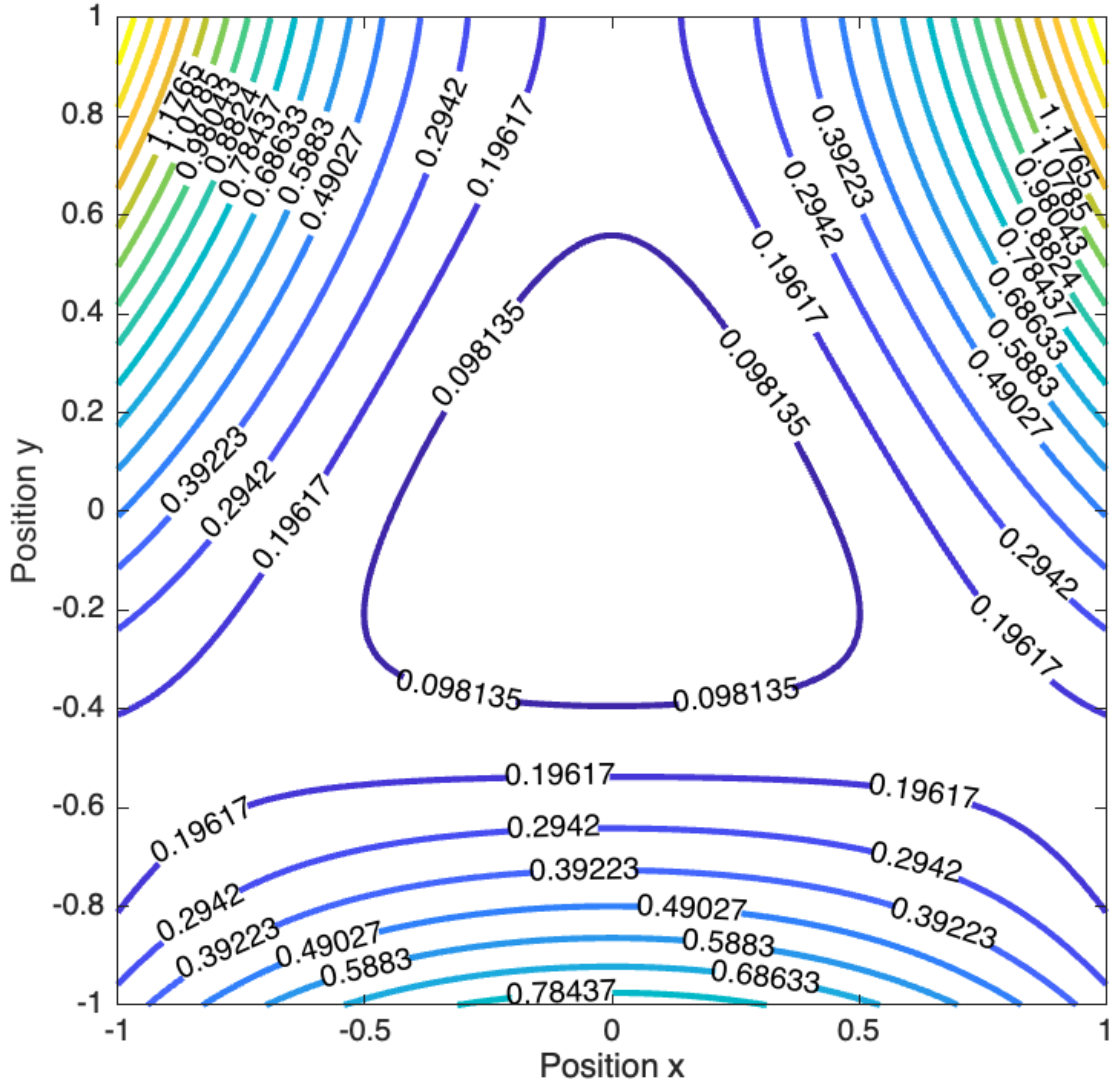
III. Henon-Heiles system

(Hamiltonian chaos)

Henon-Heiles Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + u(x, y)$$

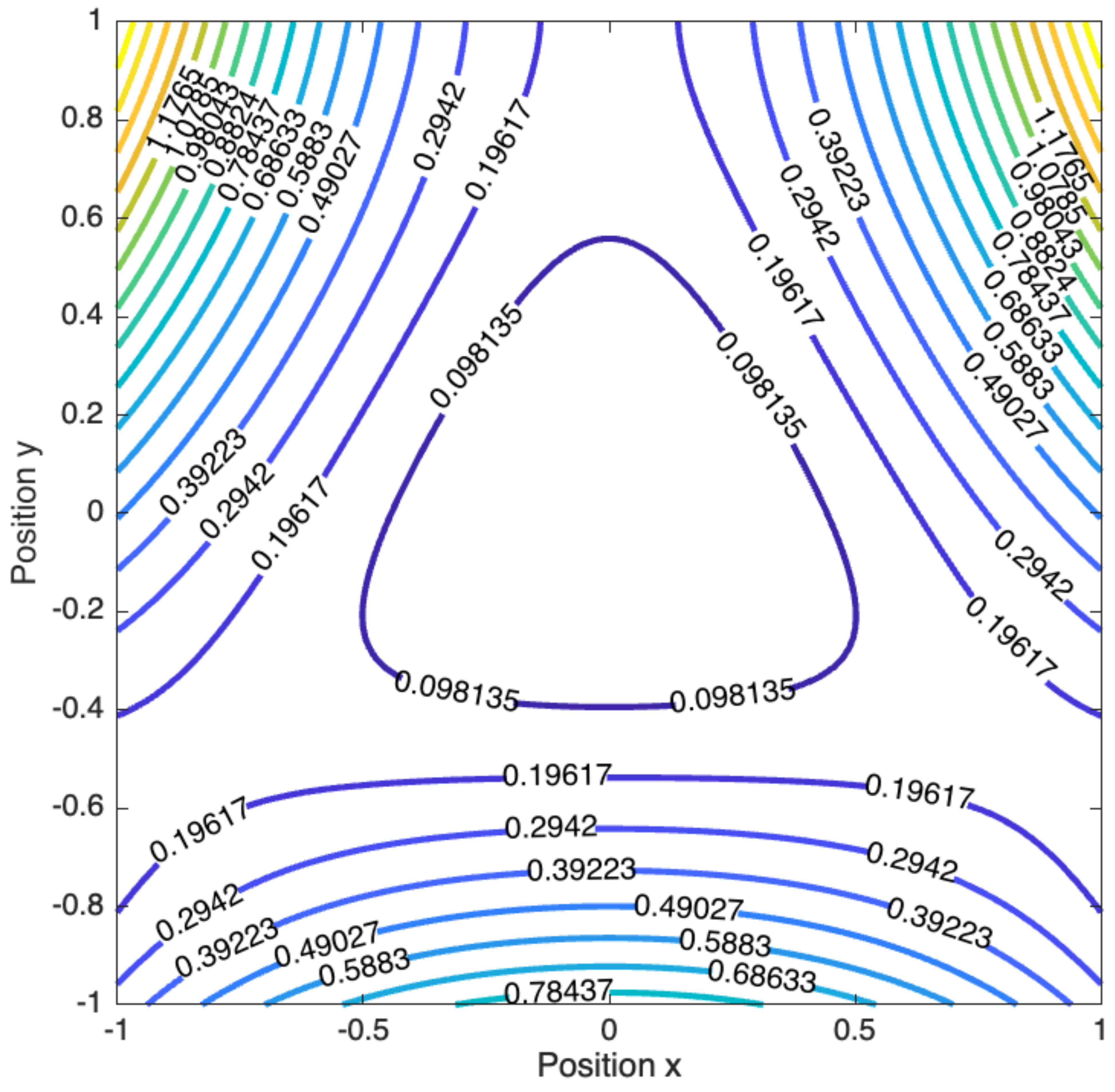
$$u = \frac{x^2 + y^2}{2} + x^2 y - \frac{y^3}{3}$$



https://mathworld.wolfram.com/Henon-HeilesEquation.html

Henon-Heiles potential energy

```
close;

xmax=1.0;

x = linspace(-xmax,xmax);
y = linspace(-xmax,xmax);
[X,Y] = meshgrid(x,y);

V = 0.5*(X.^2+Y.^2) + X.*X.*Y-(1/3)*Y.*Y.*Y;

figure(1)
[C, h]=contour(X,Y,V,16);
axis equal

xlabel 'Position x'
ylabel 'Position y'
clabel(C,h)
h.LineWidth = 2;
```



https://mathworld.wolfram.com/Henon-HeilesEquation.html

Henon-Heiles dynamics

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -x - 2xy$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_y$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -y - (x^2 - y^2)$$

conserves total energy

$$\frac{dH}{dt} = 0$$

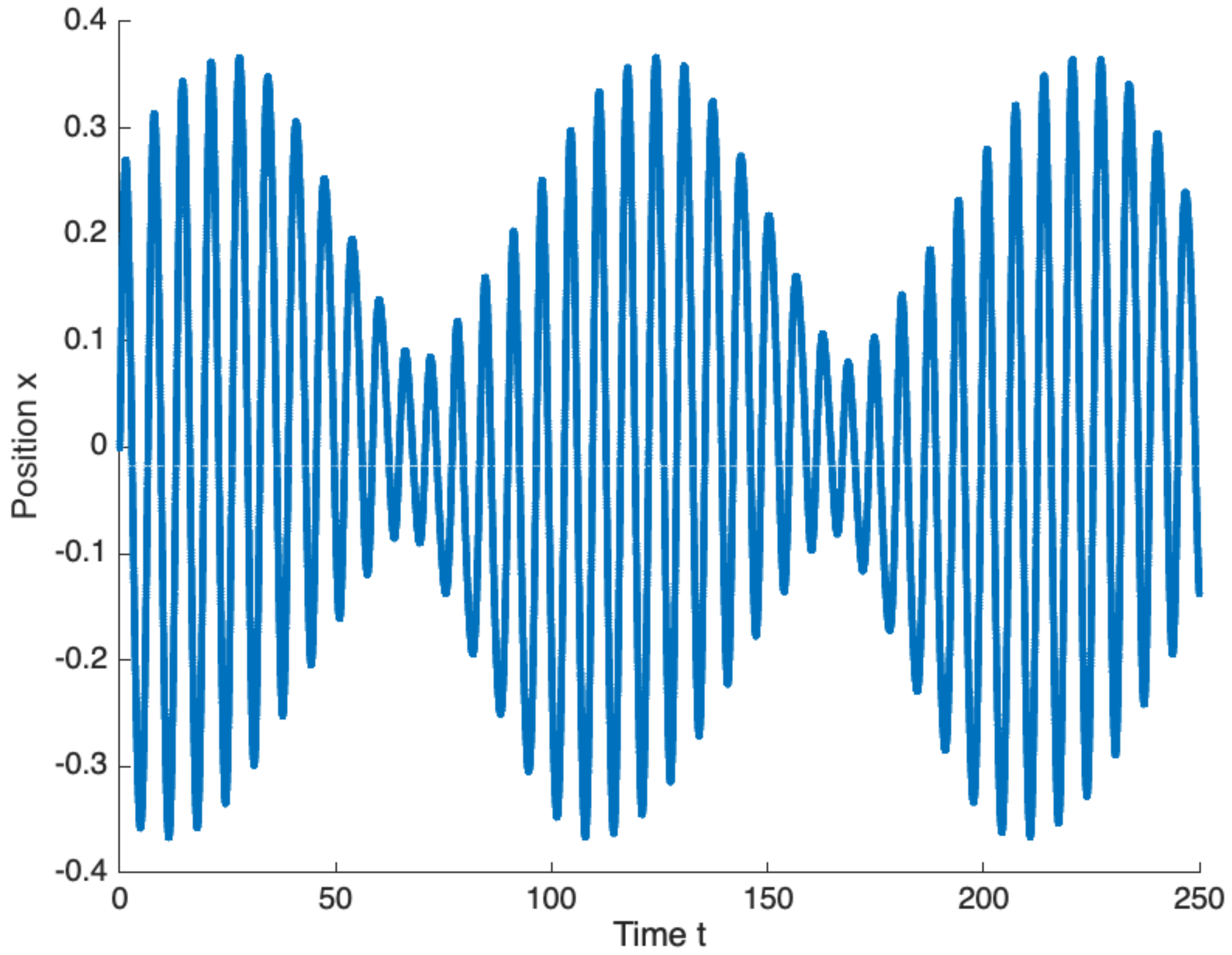
https://mathworld.wolfram.com/Henon-HeilesEquation.html

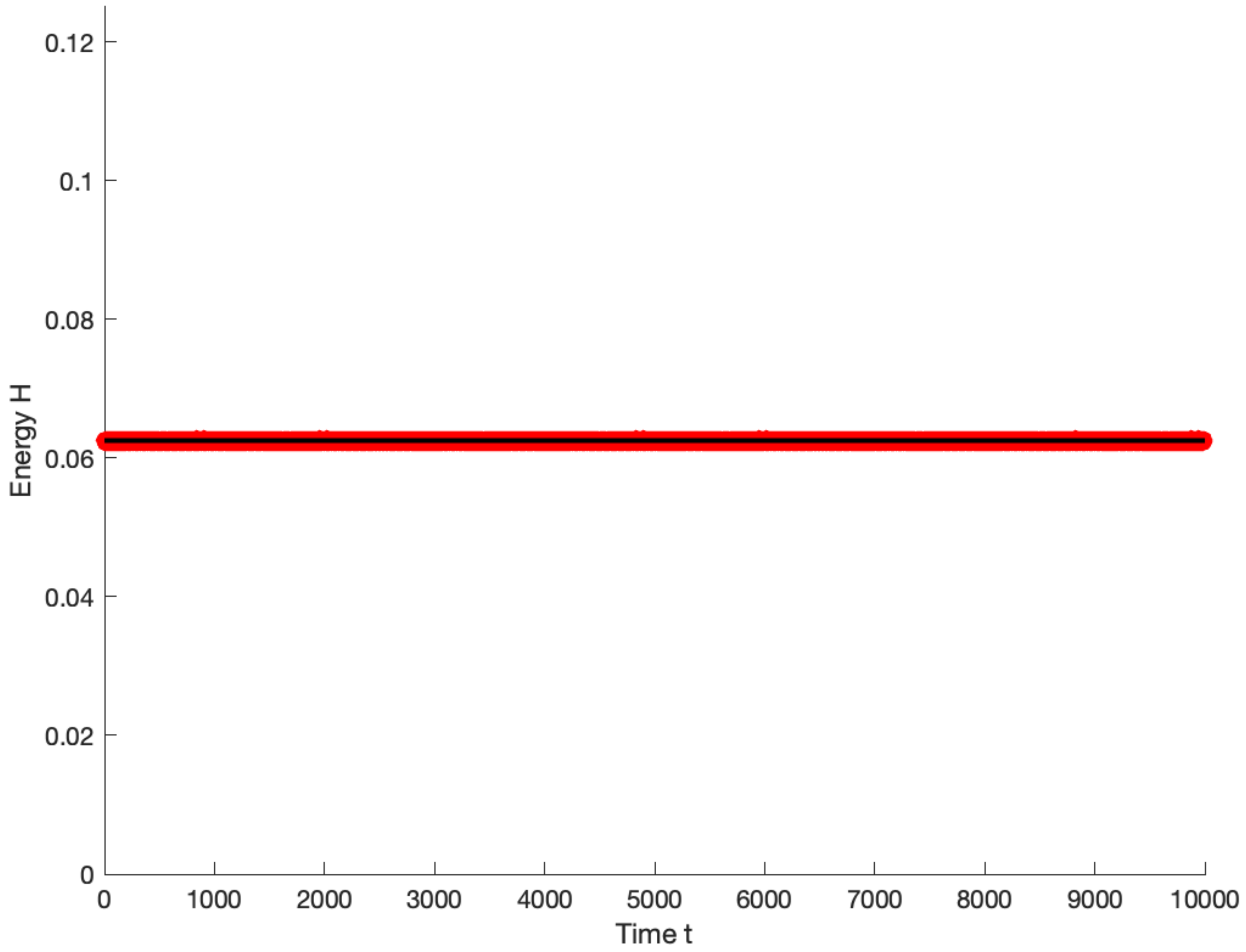
Symplectic integrator

```
for n=1:nmax

    t(n+1)  = t(n) + dt;

    px(n+1) = px(n) + dt*( -x(n) - 2*x(n)*y(n) );
    py(n+1) = py(n) + dt*( -y(n) + y(n)^2-x(n)^2);

    x(n+1)  = x(n) + dt* px(n+1);
    y(n+1)  = y(n) + dt* py(n+1);

end;
```

https://mathworld.wolfram.com/Henon-HeilesEquation.html

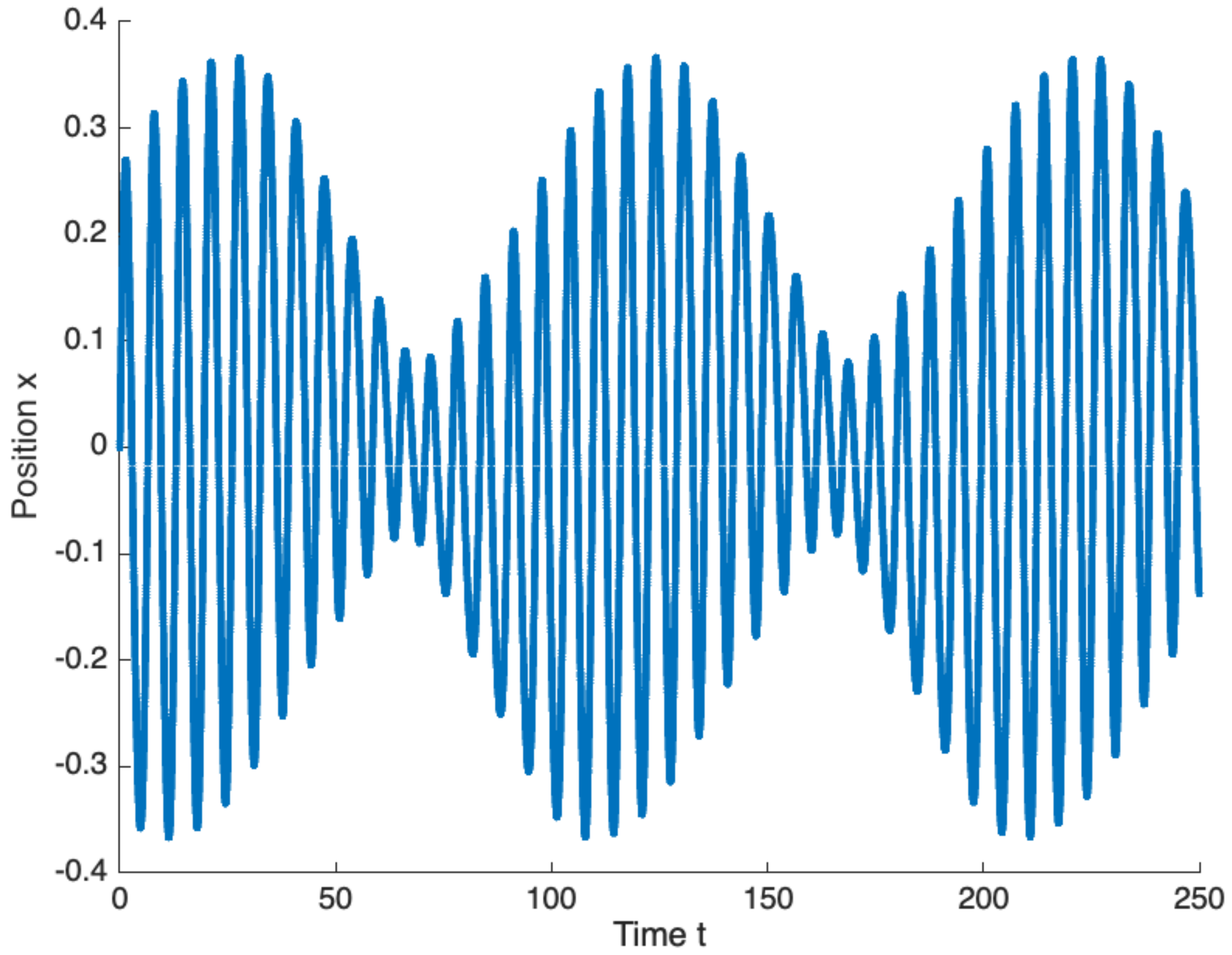
Simulation: Periodic low-energy regime

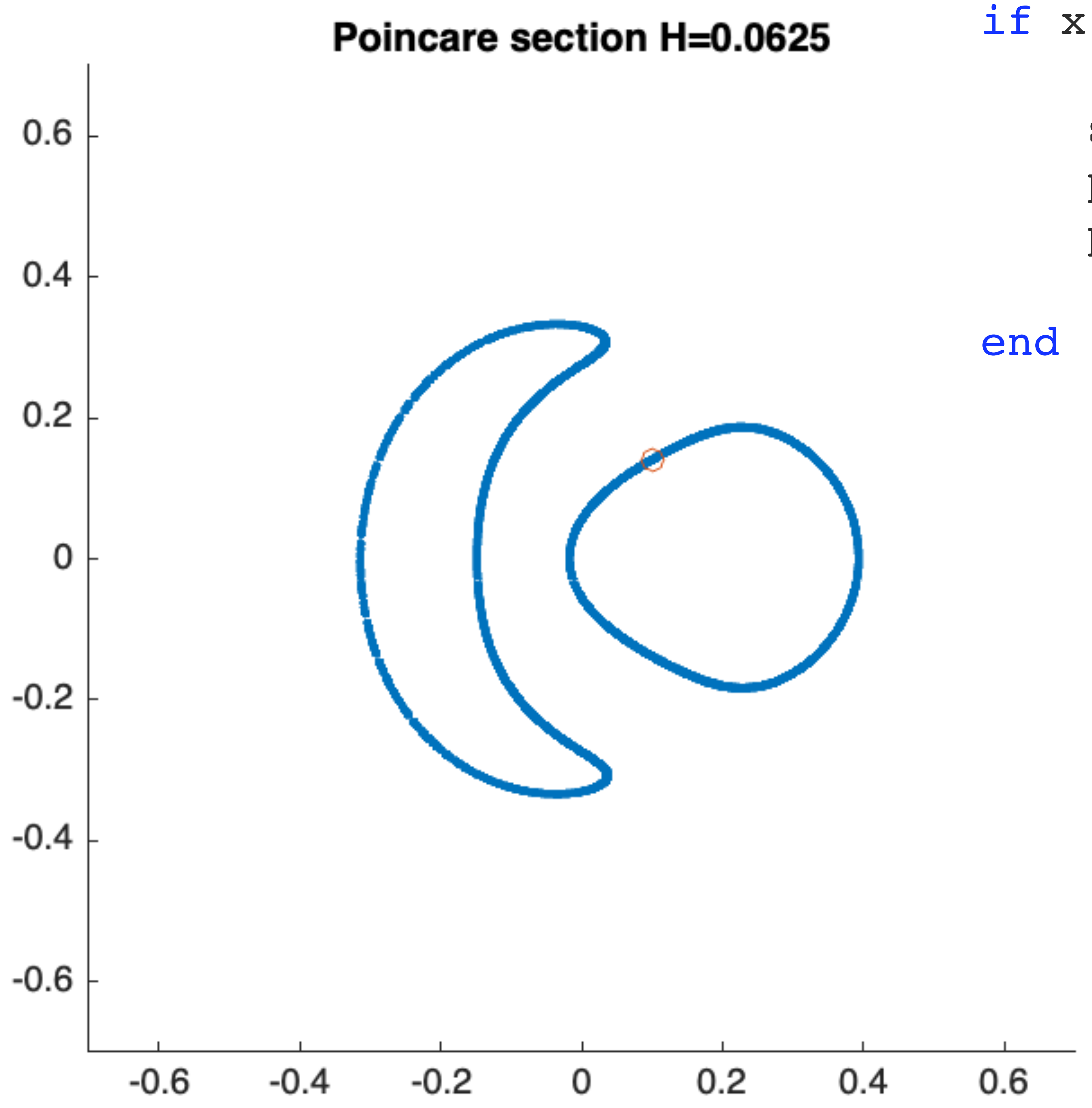




Sympletic Euler integrator with dt=0.01

Simulation: Periodic low-energy regime

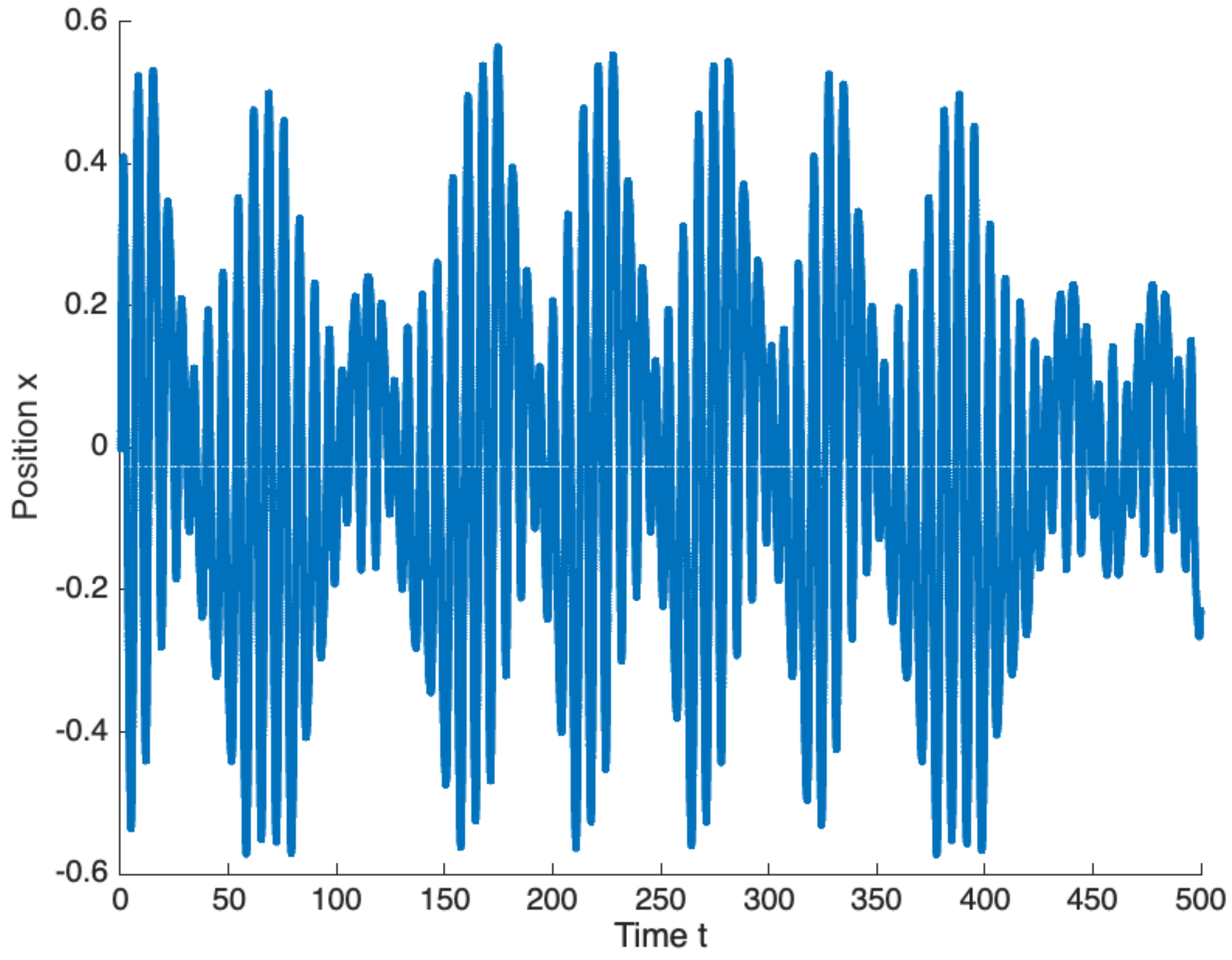


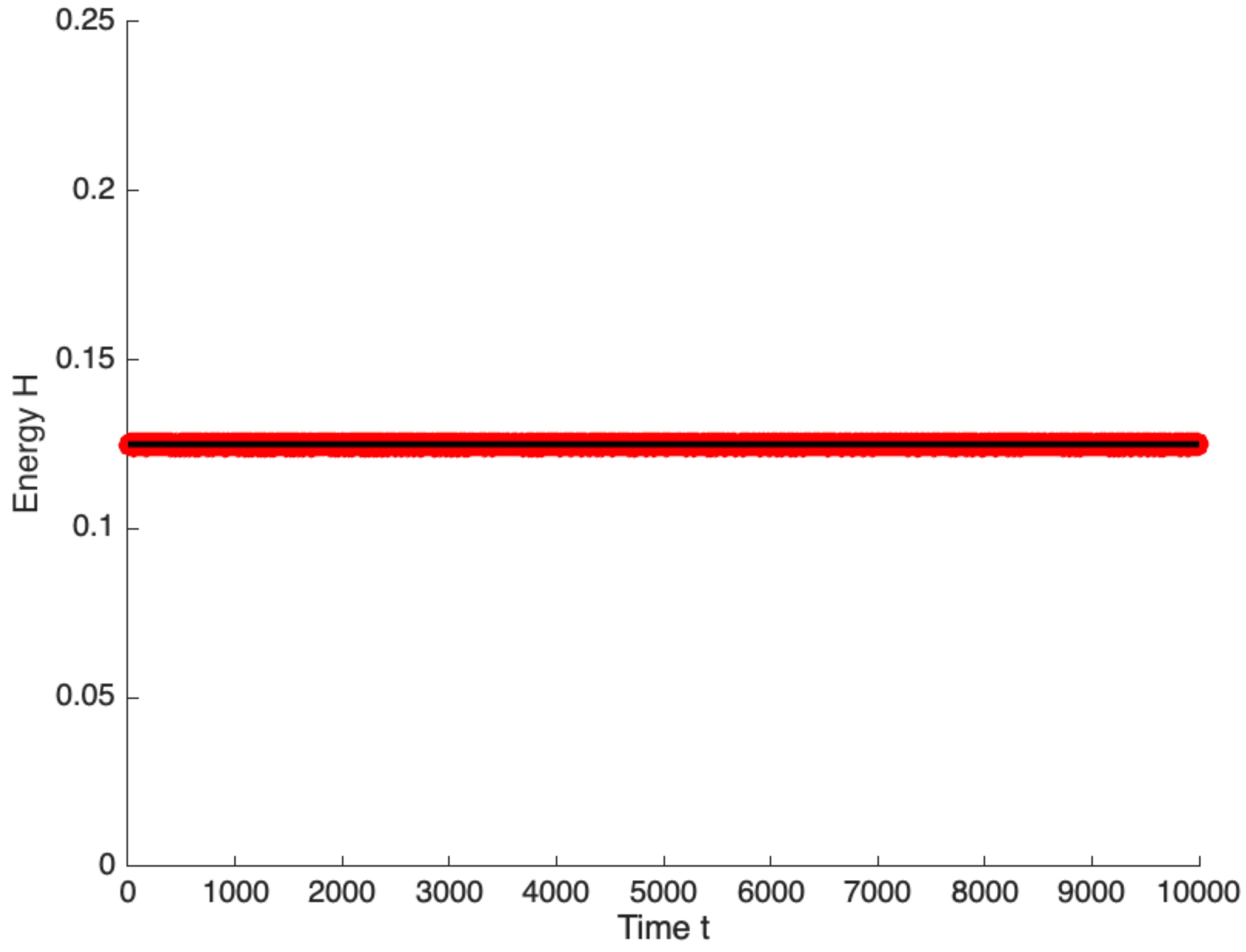



```
if x(n+1) * x(n) < 0

    s=s+1;
    pc1(s) = y(n);
    pc2(s) = py(n);

end
```

Sympletic Euler integrator with dt=0.01

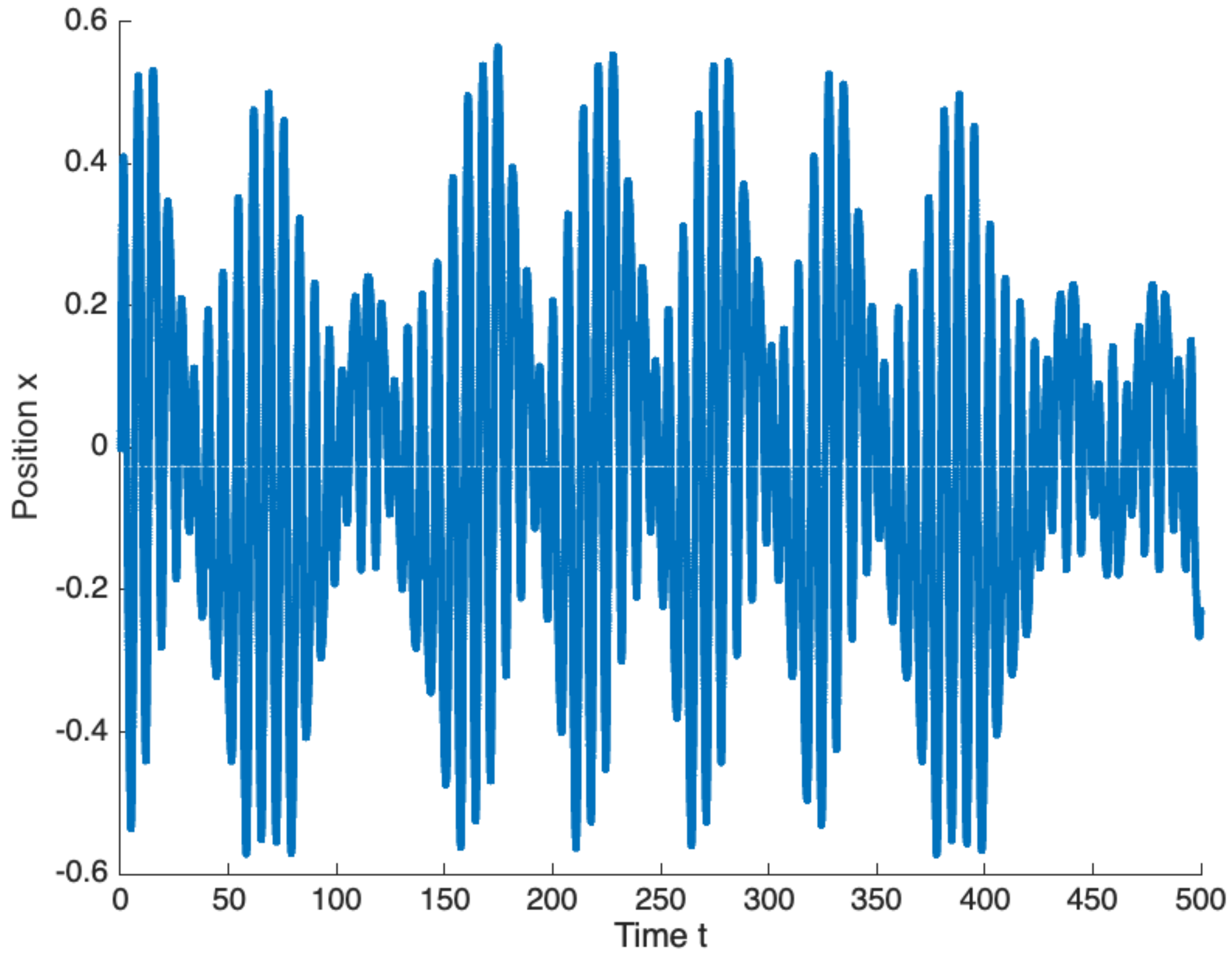
Simulation: Chaotic high-energy regime

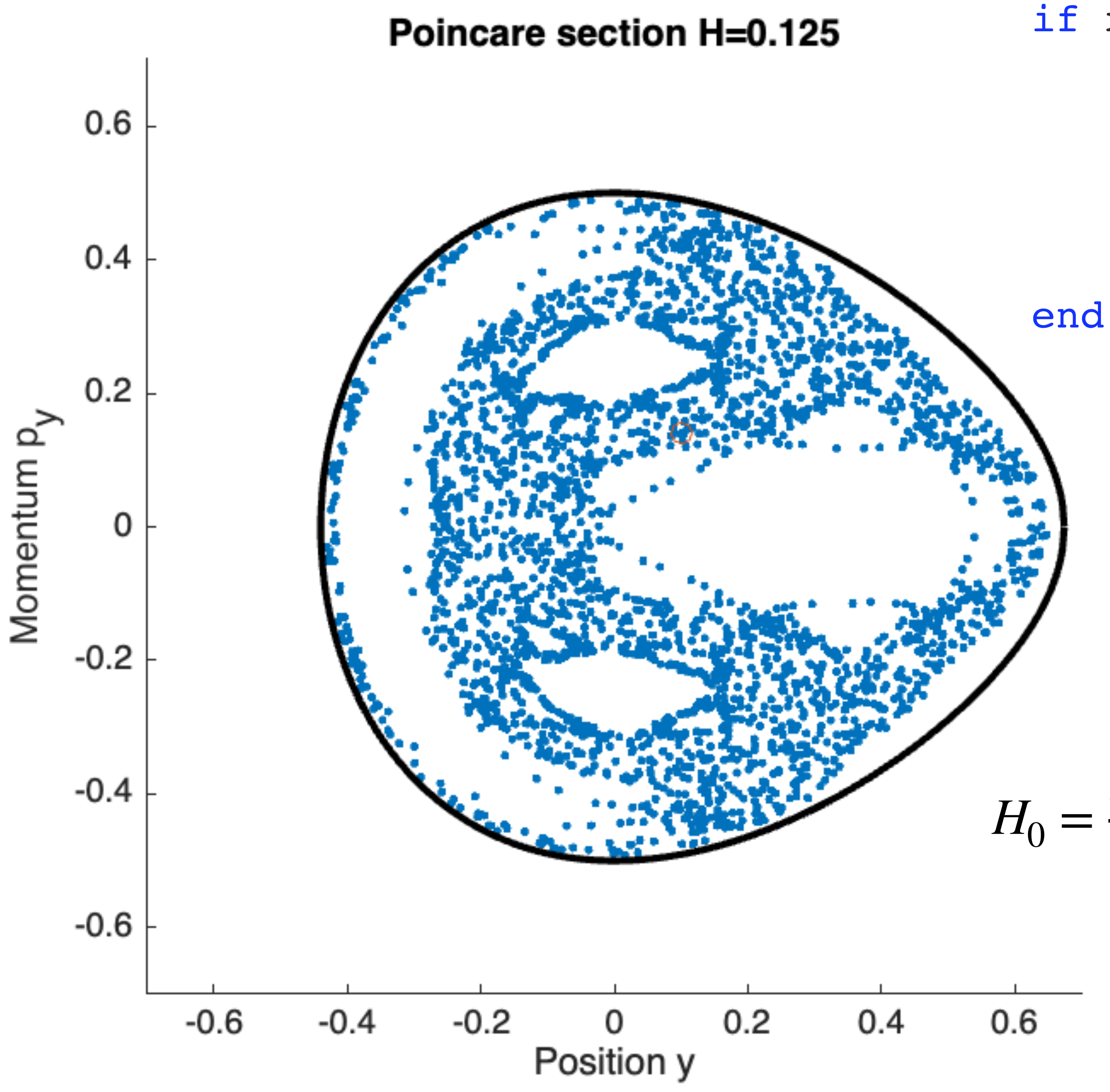




Sympletic Euler integrator with dt=0.01

Simulation: Chaotic high-energy regime






```
if x(n+1) * x(n) < 0

    s=s+1;
    pc1(s) = y(n);
    pc2(s) = py(n);

end
```

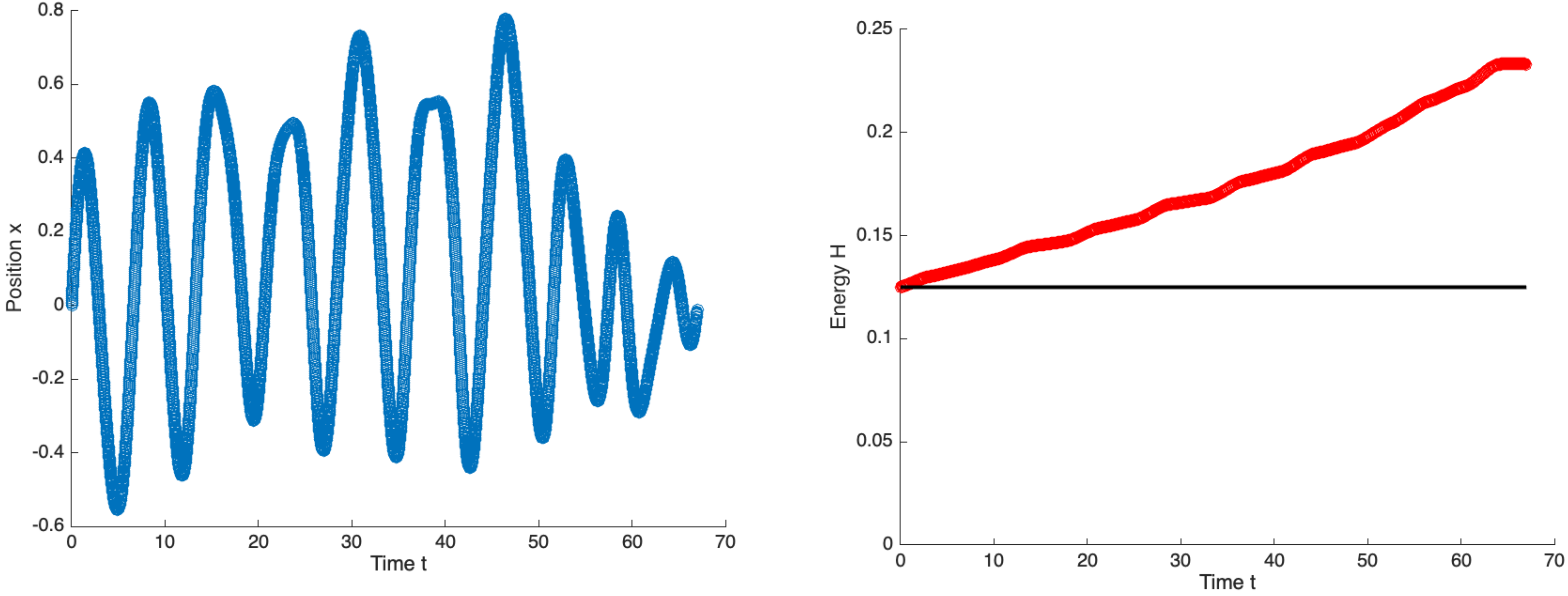
$$H_0 = \frac{p_y^2}{2} + \frac{y^2}{2} - \frac{y^3}{3}$$

Sympletic Euler integrator with dt=0.01

Non-Symplectic integrator

```
for n=1:nmax

    t(n+1)  = t(n) + dt;

    px(n+1) = px(n) + dt*( -x(n) - 2*x(n)*y(n) );
    py(n+1) = py(n) + dt*( -y(n) + y(n)^2-x(n)^2);

    x(n+1)  = x(n) + dt* px(n);
    y(n+1)  = y(n) + dt* py(n);

end;
```

Simulation: Chaotic high-energy regime



NON-sympletic Euler integrator dt=0.01