Chaos in discrete and continuous dynamical systems

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Chaos

“Impossibility to make accurate long term predictions”

Discrete dynamical systems

\[ x_{n+1} = f(x_n) \]

\[ x_n = f \circ f \circ \cdots \circ f(x_0) = f^n(x_0) \]

Continuous dynamics

Flow map / velocity field

\[ \dot{X}(t) = f(X(t)) \]

\[ X(0) = X_0 \]

System chaotic iff distance between solutions diverges rapidly even when initial conditions very close

Distance between two start values:

\[ x_0 \quad \quad \quad \quad \quad \quad \quad x_0 + \varepsilon \]

After \( N \) iterations:

\[ f^N(x_0) \quad \quad \quad \quad \quad \quad \quad f^N(x_0 + \varepsilon) \]

Maximal Lyapunov exponents

\[ \| \delta(t) \| \approx \| \delta(0) \| e^{\lambda t} \]

https://en.wikipedia.org/wiki/Lyapunov_exponent
I. Logistic map
Discrete dynamics

Basic exponential growth model

\[ x_{n+1} - x_n = gx_n \]

\[ x_{n+1} = (1 + g)x_n = rx_n \]

\[ \Rightarrow x_n = (1 + g)^nx_0 = r^n x_n \]

(Slightly) more realistic model including competition for resources

Logistic map

\[ x_{n+1} = r(x_n - x_n^2) = rx_n(1 - x_n) \]

Fixed points

General definition

\[ x_n = f(x_n) \]

For logistic map

\[ x_n = rx_n(1 - x_n) \]

1. FP \( x_- = 0 \)

"Extinction"

2. FP \( x_n \neq 0 \)

\[ 1 = r(1 - x_n) \]

Makes sense for \( r > 1 \)

\[ x_+ = 1 - \frac{1}{r} = \frac{r - 1}{r} \]
Cobweb plots

\[ x_{n+1} = x_n + 1 \]

only FP and stable

\[ 0 \leq r \leq 1 \]

\[ x_1 = 0 \]

Cobweb plots

\[ x_{n+1} = r x_n (1 - x_n) \]

for different values of \( r \). When \( 1 \leq r \leq 2 \), the system has a stable fixed point (FP) and a non-oscillatory approach.

Stable FP: \( x_+ = \frac{r - 1}{r} \)

Cobweb plots

\[ x_{n+1} = r - 1 \]

\[ 2 \leq r \leq 3 \]

\[ x_+ = \frac{r - 1}{r} \]

stable FP
oscillatory approach

Cobweb plots

\[ x_{n+1} \]

\[ 3 \leq r < 1 + \sqrt{6} \approx 3.44949 \]

Steady-state oscillations between two values

Cobweb plots

3.44949 \lesssim r \lesssim 3.54409

"Period doubling"
Steady-state oscillations
Among four values

Cobweb plots

$3.54409 \lesssim r \lesssim 3.56995$

“Period doubling cascade”

Steady-state oscillations among 8, 16, ... values

Cobweb plots

\[ x_{n+1} \]

\[ r \geq 3.56995 \]

"Chaos"

Feigenbaum plot

Homework 1: write a computer program that creates this plot
Lyapunov exponents

\[ \lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \]
II. Lorenz system
Lorenz system: dynamics

\[ \dot{x} = \sigma (y - x) \]
\[ \dot{y} = x (\rho - z) - y \]
\[ \dot{z} = xy - \beta z \]
Fixed points & linear stability

1. FP
   \[ x = y = z = 0 \]

2. FP
   \[ z = \rho - 1 \]
   \[ x = y = \pm \sqrt{\beta(\rho - 1)} \]

Homework 2: Evaluate the linear stability of these fixed points
Hodge-Helmholtz decomposition

\[
\dot{x} = f_1 = \sigma(y - x)
\]

\[
\dot{y} = f_2 = x(\rho - z) - y
\]

\[
\dot{z} = f_3 = xy - \beta z
\]

\[f = G + R\]

Gradient field

\[G = - \begin{pmatrix} \sigma x \\ y \\ \beta z \end{pmatrix}\]

Rotation fields

\[R = \begin{pmatrix} \sigma x \\ x(\rho - z) \\ xy \end{pmatrix}\]

\[\nabla \cdot R = 0\]

\[G = - \nabla \Phi\]

\[\Phi = \frac{1}{2} (\sigma x^2 + y^2 + \beta z^2)\]
% Solve over time interval [0,100] with initial conditions [1,1,1]

% ''f'' is set of differential equations
% ''a'' is array containing x, y, and z variables
% ''t'' is time variable

sigma = 10;
beta = 8/3;
rho = 28;

f = @(t,a) [-sigma*a(1) + sigma*a(2); rho*a(1) - a(2) - a(1)*a(3); -beta*a(3) + a(1)*a(2)];
[t,a] = ode45(f,[0 100],[1 1 1]); % Runge-Kutta 4th/5th order ODE solver
Simulation

plot(t,a(:,1))
xlabel('t')
ylabel('x')

plot(t,a(:,2))
xlabel('t')
ylabel('y')

plot(t,a(:,3))
xlabel('t')
ylabel('z')
Simulation

plot3(a(:,1),a(:,2),a(:,3))
xlabel('x')
ylabel('y')
zlabel('z')
III. Henon-Heiles system

(Hamiltonian chaos)
Henon-Heiles Hamiltonian

\[ H = \frac{p_x^2 + p_y^2}{2} + u(x, y) \]

\[ u = \frac{x^2 + y^2}{2} + x^2 y - \frac{y^3}{3} \]

Henon-Heiles potential energy

close;
xmax=1.0;
x = linspace(-xmax,xmax);
y = linspace(-xmax,xmax);
[X,Y] = meshgrid(x,y);

V = 0.5*(X.^2+Y.^2) + X.*X.*Y-(1/3)*Y.*Y.*Y;

figure(1)
[C, h]=contour(X,Y,V,16);
axis equal

xlabel 'Position x'
ylabel 'Position y'
clabel(C,h)
h.LineWidth = 2;
Henon-Heiles dynamics

\[
\begin{align*}
    \dot{x} &= \frac{\partial H}{\partial p_x} = p_x \\
    \dot{y} &= \frac{\partial H}{\partial p_y} = p_y \\
    \dot{p}_x &= -\frac{\partial H}{\partial x} = -x - 2xy \\
    \dot{p}_y &= -\frac{\partial H}{\partial y} = -y - (x^2 - y^2)
\end{align*}
\]

conserves total energy

\[
\frac{dH}{dt} = 0
\]
Symplectic integrator

for n=1:nmax

    t(n+1)  = t(n) + dt;

    px(n+1) = px(n) + dt*( -x(n) - 2*x(n)*y(n) );
    py(n+1) = py(n) + dt*( -y(n) + y(n)^2-x(n)^2);

    x(n+1)  = x(n) + dt* px(n+1);
    y(n+1)  = y(n) + dt* py(n+1);

end;

Simulation: Periodic low-energy regime

Symplectic Euler integrator with $dt=0.01$
Sympletic Euler integrator with \( dt=0.01 \)

Simulation: Periodic low-energy regime

\[
\text{if } x(n+1) \times x(n) < 0 \\
\text{ \quad } s = s + 1; \\
\text{ \quad } pc1(s) = y(n); \\
\text{ \quad } pc2(s) = py(n); \\
\text{end}
\]
Simulation: Chaotic high-energy regime

Symplectic Euler integrator with $dt=0.01$
Symplectic Euler integrator with $dt=0.01$

$H_0 = \frac{p_y^2}{2} + \frac{y^2}{2} - \frac{y^3}{3}$

Simulation: Chaotic high-energy regime

if $x(n+1) \times x(n) < 0$

$s = s+1;$

$pc1(s) = y(n);$  
$pc2(s) = py(n);$  

end
Non-Symplectic integrator

for n=1:nmax

    t(n+1) = t(n) + dt;
    px(n+1) = px(n) + dt* ( -x(n) - 2*x(n)*y(n) );
    py(n+1) = py(n) + dt* ( -y(n) + y(n)^2-x(n)^2);
    x(n+1) = x(n) + dt* px(n);
    y(n+1) = y(n) + dt* py(n);

end;
Simulation: Chaotic high-energy regime

NON-sympletic Euler integrator  $dt=0.01$