Curvature, cartography and pizzas slices

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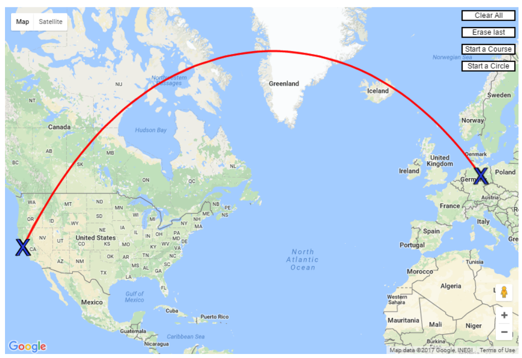
1 Cartography

From the earliest maps of antiquity to Google Maps, cartography has continually reinvented itself through geographic, technological, and even mathematical discoveries.



A map is a correspondence between positions on the Earth's surface and points on a sheet of paper. We expect it to allow us to orient ourselves and thus preserve the geographic or geometric properties of the area we wish to map. Here, we will study the construction of *faithful* maps, that is, maps that preserve distances (modulo a change of scale).

To *model* this construction, we will provide the basic definitions of *metric geometry* and *geometry of surfaces*. We will see that this problem is in fact related to one of the most fundamental results of modern geometry, the famous Theorema Egregium of Carl Friedrich Gauss (1777-1855), presumably stated in 1828.

We will particularly understand why airplanes do not follow straight lines on the maps we know and why the sizes of countries do not correspond to reality. We will see what choices the maps we use reflect.

As a bonus, we will explain the shape of certain constructions and why the best way to eat a slice of pizza is to fold it lengthwise.

1.1 Mapping the Earth's Surface

What we expect from a map at first is that each position on our map corresponds to a unique point on Earth.

In mathematical terms, this means that a map must be a **bijection** from the Earth's surface to our sheet of paper. But there are many bijections that do not preserve the Earth's geometry at all.

1.2 Faithful Maps

What we expect from a **faithful** map is that it preserves distances. However, as we deal with **intrinsic** geometry on the sphere **immersed** or **embedded** in three-dimensional ambient space, we must be precise about what we mean by distance. Indeed, if we consider the cities of Paris and Sydney, we must be precise in what we call the distance between these cities, it can be the distance in the ambient space (thus crossing the interior of the Earth), the "as the crow flies" distance on Earth, the minimal travel time between the two cities according to Google Maps...

2 Metric Spaces

2.1 Distance

Definition The intuitive notion of distance can therefore correspond to quite different things, and we thus need a definition loose enough to include all these possible ideas:

Definition 2.1 (Distance)**.** *A distance d on a set E is an application that associates a positive number with two points of the set and satisfies the following properties:*

For any choice of three elements a, b, and c of E, we have:

- *SEPARATION: The distance between two points is zero if and only if the two points are coincident: If $a \neq b$, then,*
$$d(a, b) > 0,$$
and,
$$d(a, a) = 0.$$
- *SYMMETRY: The distance between a point a and a point b is the same as the distance between a point b and a point a.*
$$d(a, b) = d(b, a).$$
- *TRIANGLE INEQUALITY: The distance from a to c passing through b is at least as great as the distance between a and c.*
$$d(a, c) \leqslant d(a, b) + d(b, c).$$

Definition 2.2 (Metric Space)**.** *A set equipped with a distance between its points is a* **metric space***.*

If the set we are considering is called X and the distance we are considering on it is called d, we will denote the metric space (X, d).

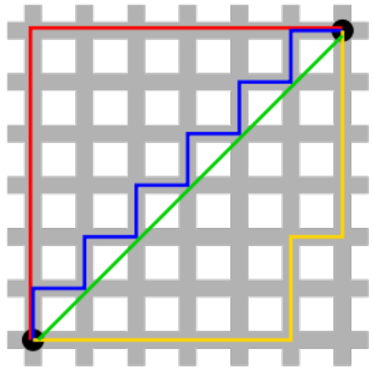
Examples of Distances Let's start with the most usual distance:

Example 2.1 (Euclidean Plane and Space)**.** *Given two points either in the plane or in space, the length of the segment connecting them is indeed a distance.*

Remark 2.1. *This distance can also be referred to as L^2 distance. It's the distance we are accustomed to in our daily lives, and it is in this context, and only in this context, that we have results such as Pythagoras' and Thales' theorems.*

Exotic Distances of the Plane

Example 2.2 (Distance in a Typical American City: L^1, Manhattan Distance, Taxicab Geometry,...)**.** *In a typical large American city, the streets (east-west) and avenues (north-south) form a grid of the city, and the walking or driving distance is actually the sum of the number of avenues and streets crossed. The points equidistant from a fixed point then form a diamond shape.*



Example 2.3 (French Distance)**.** *A typically French distance, d_{SNCF}: suppose a country is so centralized that all satisfactory train lines of the country pass through a given city, which we will call P(aris). The distance by train between two cities is 1) the usual distance if they are on the same (train) line, e.g. $d_{SNCF}(Angoulème, Tours) = d_{Euc}(Angoulème, Tours)$ or 2), the distance from the first city to P plus the distance from P to the second city, e.g. $d_{SNCF}(Bordeaux, Lyon) = d_{Euc}(Bordeaux, Paris) + d_{Euc}(Paris, Lyon)$.*

Intrinsic distance

Example 2.4 (Geodesic Distance on a Surface in Space)**.** *If a surface Σ is embedded in space, we can then define an* intrinsic *distance on the surface corresponding to the length of the shortest path remaining on the surface. This is the* **geodesic** *distance:*

If x and y are points of the surface Σ, we can consider the infimum of the lengths of curves included in Σ joining x and y.

The curves of minimal length between two points (when they exist) are called geodesics.

This is the **intrinsic** *or* **geodesic** *distance on the surface.*

Remark 2.2. *This last example is what motivates the definition of an intrinsic geometry of the surface, as the ambient space does not influence the calculation of these distances.*

Remark 2.3. *When we commonly talk about the distance between Paris and Sydney, we think of the distance on the surface of the Earth, which is the geodesic distance (hence the name), we do not think of the distance in the ambient space (minimized by a segment passing through the Earth).*

Remark 2.4. *There are many other distances that can be induced on concrete sets that are used daily. For example, one can talk about the distance from one internet page to another by considering the number of clicks needed to go from one to the other by following links. One can consider the distance from one person to another by considering the number of steps to go from one to the other, these steps can be very diverse: moving from a friend to a friend of a friend, from a co-author to the co-author of a co-author, etc...*

2.2 Isometry

A faithful map of the Earth or one of its sub-parts is actually a correspondence between points on the Earth's surface and points on a sheet of paper that preserves distances (up to a scale change...).

In more technical terms, such a correspondence is called an *isometry*.

Definition 2.3 (Isometry)**.** *An isometry between two metric spaces is a way of matching the points of the two spaces that preserves the distances between these points.*

That is, if (X_1, d_1) and (X_2, d_2) are two metric spaces, an application f associating each element of X_1 with an element of X_2 is an isometry if for all points x_1 and y_1 of X_1, we have:

$$d_1(x_1, y_1) = d_2(f(x_1), f(y_1)).$$

Definition 2.4 (Faithful Map)**.** *We will therefore call a* **faithful map** *an isometry between a sub-part of the Earth and the Euclidean plane.*

3 Smooth Surfaces of Space

Let's consider the modeling of what the Earth's surface is. The goal is to define a mathematical object having the same properties as the Earth's surface while discarding *pathological* behaviors that would have nothing to do with the intuition of what we would like to call a surface of space.

If we do not limit what we mean by a surface of space, we can end up with infinitely complicated objects (see the course on fractals for example) which we do not wish to consider.

The main observation that will allow us to limit unexpected behaviors is the following:

It was long believed that the Earth was flat!

This is natural, when we move, the ground on which we walk seems flat and the horizon distant. At our scale, the geometry that is valid is that of the plane, where the theorems of Pythagoras and Thales are true.

The Earth is so much larger than the distances we are used to (the Earth has a circumference of 40,000km) that, by comparison, we live in a sort of *zoom* on the Earth's surface which would be a plane.

Once we look from farther away, we begin to observe that the Earth is curved although it seems flat on a small scale:

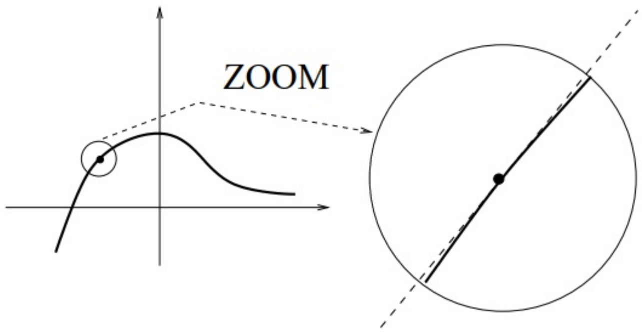
This observation is the basis of the definition of smooth surfaces: on a small scale, they resemble a plane. This simple property greatly limits the objects that the definition will encompass while being consistent with our experience of reality.

3.1 Definition

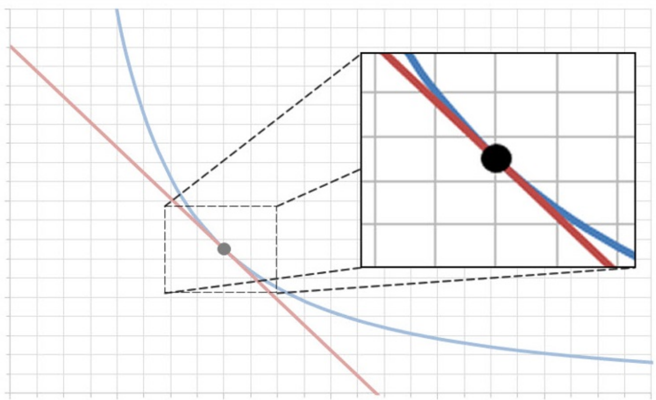
A surface is to the plane what a curve is to a straight line. Let's start by defining what we mean by *smooth* in the context of curves:

Curves

Definition 3.1 (Smooth Curve)**.** *A* **smooth curve** *is a set of points that resembles a straight line around each of its points if we zoom in sufficiently.*



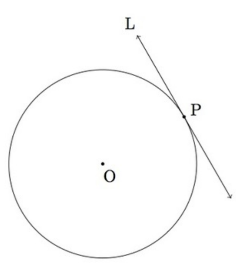
At each point, we can associate a straight line, called **tangent***, which best approximates the curve on a small scale. It is in fact the one that appears when we zoom around the point, it corresponds to the direction of the velocity that an object moving on the curve would have.*



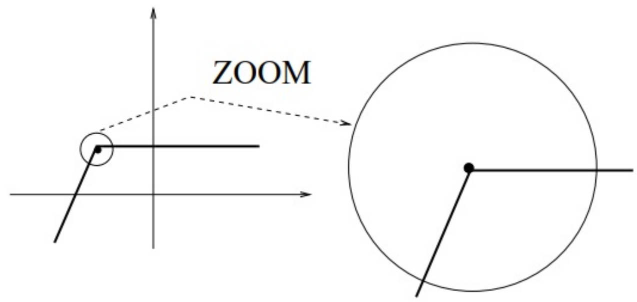
Remark 3.1. *The objective of this definition is to eliminate* pathological behaviors *and objects that are sometimes called* mathematical monsters *that do not correspond to what we would commonly call a* curve*.*

Examples and Non-Examples

1. A straight line is a smooth curve.
2. A circle is a smooth curve.



3. A triangle, a square, and more generally all polyhedra are not smooth curves. Indeed, if we consider a ***vertex*** (a point where two ***sides*** meet at an angle), we can zoom as much as we want, the angle is always present and we will never be close to a straight line.



Remark 3.2. *However, they are smooth curves* in pieces. *Indeed, apart from the points where there are angles, we can always zoom and define a tangent line.*

4. More generally, anything that has points or angles will not be considered smooth.

Surfaces

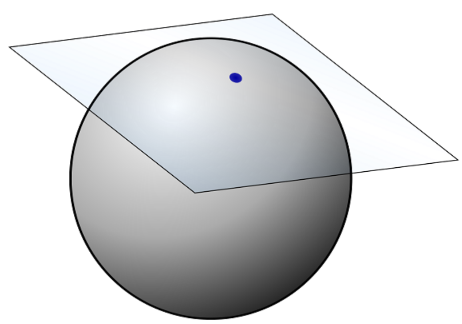
Definition 3.2 (Smooth Surface)**.** *A* **smooth surface** *is a set of points that resembles a plane around each of its points if we zoom in sufficiently.*

At each point, we can associate a plane, called **tangent***, which best approximates the surface on a small scale. It is in fact the one that appears when we zoom around the point. This plane is exactly composed of the tangent lines to all the curves of the surface passing through the considered point.*

Remark 3.3. *The definition again aims not to include complicated and counterintuitive objects.*

Examples and Non-Examples

1. A plane is a smooth surface.
2. A sphere is a smooth surface.



3. Anything that has angles, points will not be a smooth surface, such as cubes and other polyhedra, for example.

Some of the main examples of smooth surfaces are *nondegenerate* (i.e. without critical points) **level sets** of smooth maps: $\phi\mathbb{R}^3 \to \mathbb{R}$:

$$\Sigma_a = \{x, \phi(x) = a\}$$

is a smooth surface for $a \in \mathbb{R}$ so that $\nabla\phi = (\partial_x\phi, \partial_y\phi, \partial_z\phi) \neq 0 \in \mathbb{R}^3$ on Σ_a. The **tangent space** at (x_0, y_0, z_0) in that case is the orthogonal of the gradient $\nabla\phi(x_0, y_0, z_0)$.

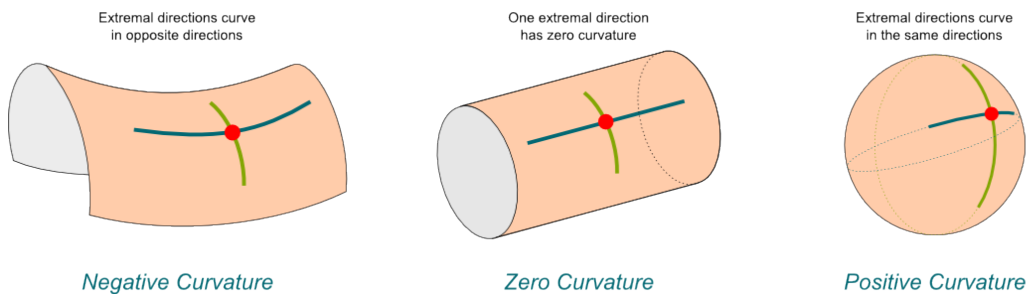
Example 3.1. *the sphere of radius $R > 0$ is the level set Σ_{R^2} of the smooth function $\phi : (x, y, z) \mapsto x^2 + y^2 + z^2$ with $\nabla\phi(x, y, z) = 2(x, y, z) \neq 0$.*

4 Curvature

Curvature is a mathematical concept whose formal version is extremely complicated. Intuitively, however, it is a very natural quantity that allows us to measure to what extent the surface of interest is not flat.

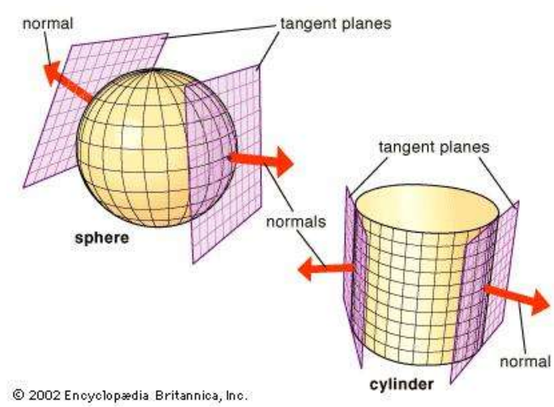
4.1 Gauss Curvature: Curvature and Position Relative to the Tangent Plane

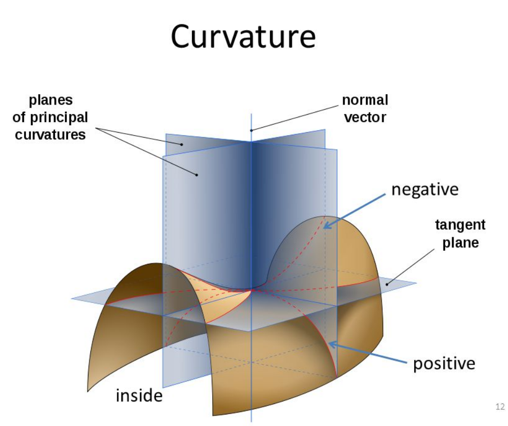
Gauss curvature is the product of the minimal and maximal curvatures of geodesics starting from a point.



Thus:

- a surface entirely contained on one side of its tangent plane will have positive curvature,
- a surface passing from one side to the other of its tangent plane will have negative curvature,
- a surface will have zero curvature if and only if one of its segments is contained in the tangent plane.





4.2 Toponogov's Theorem: Curvature and the Shape of Triangles

Curvature is often a data too complicated, so it is important to understand it differently. The most fruitful way to understand curvature is by *comparison*. By comparison, we mean comparing the geometry of the surface with the geometry of the plane.

We will see that *positive* curvature corresponds to having triangles thicker than in the plane, which will have zero curvature (negative curvature will correspond to thinner triangles).

Remark 4.1. *This type of definition of curvature is often classified as* synthetic*, meaning defined through geometric properties as opposed to* analytic*, meaning defined from equations.*

Geodesic To compare triangles, it is necessary to define what is meant by a triangle on a surface. A *triangle* in the plane is the union of three segments connecting three points. For a surface, the definition is the same, but it is necessary to define what will generalize the notion of a segment, this is the *geodesic*, which, as its name suggests, was introduced to study the surface of the Earth.

Definition 4.1 (Minimizing Geodesic)**.** *A* **minimizing geodesic** *is a curve of minimal length between its starting point and its endpoint (among curves having the same starting and ending points).*

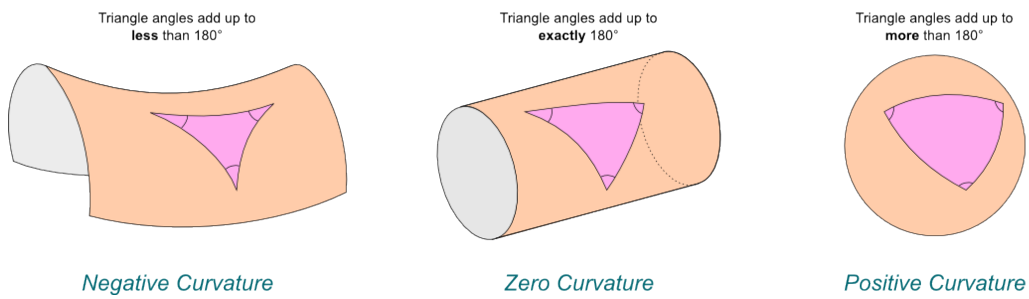
Remark 4.2. *If we move straight without deviating on a surface, we are actually following a geodesic. For example, on Earth, going straight, we follow a* great circle *that goes around the planet.*

One of the peculiarities of Euclidean geometry of the plane is that the sum of the angles of a triangle always makes 180 ° (or π radians).

Toponogov's Theorem

Theorem 4.3 (Toponogov)**.** *We have the following results/definitions for surfaces with positive or negative curvature:*

- *A surface with negative curvature has triangles thinner than in the plane (sum of angles less than* 180°*).*
- *A surface with positive curvature has triangles thicker than in the plane (sum of angles more than* 180°*).*



Remark 4.4. *A remarkable property is that the definitions of curvature by triangles or curvatures of curves are equivalent.*

5 Theorema Egregium of Gauss

One of the greatest and most beautiful theorems of surface geometry is Gauss's Theorema Egregium. This theorem gives full meaning to the notion of curvature for surfaces and shows that Gauss's curvature, which is initially only an extrinsic quantity, is in fact an intrinsic quantity that, moreover, completely determines the intrinsic geometry of the surface.

Theorem 5.1 (Theorema Egregium of Gauss)**.** *If two surfaces are isometric, then they have the same curvature.*

A consequence of this theorem is a definitive answer to the question "How to construct a faithful map of a part of the Earth?":

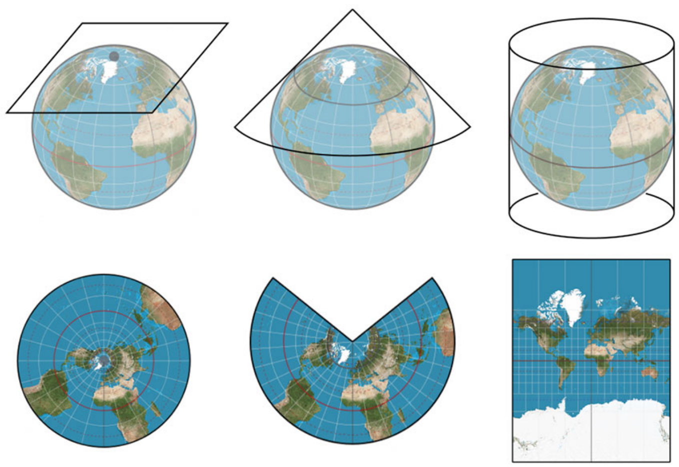
It's impossible!

Corollary 5.2. *There is no* faithful *map of non-zero area parts of the Earth.*

Indeed, the surface of the Earth (assimilated to a sphere or an ellipsoid) has a strictly positive curvature and therefore, according to the theorema egregium, if one of its parts were isometric to a part of the plane, then its curvature would be zero.

6 What Can Then Be Preserved?

So, there is no *faithful* map of the Earth, nor even of continents or countries, but there are still world maps that we are accustomed to. How were they chosen? What properties of the Earth's surface do they preserve? What choices do they reflect?



6.1 Preserving Angles

One first piece of good news is that angles can be preserved, this gives what is called a ***conformal*** map. This is a result due to Gauss, which is still today an important and difficult result of the theory.

However, a conformal map will never preserve the sizes of countries and continents, whether in terms of distances or areas.

6.2 Preserving Areas

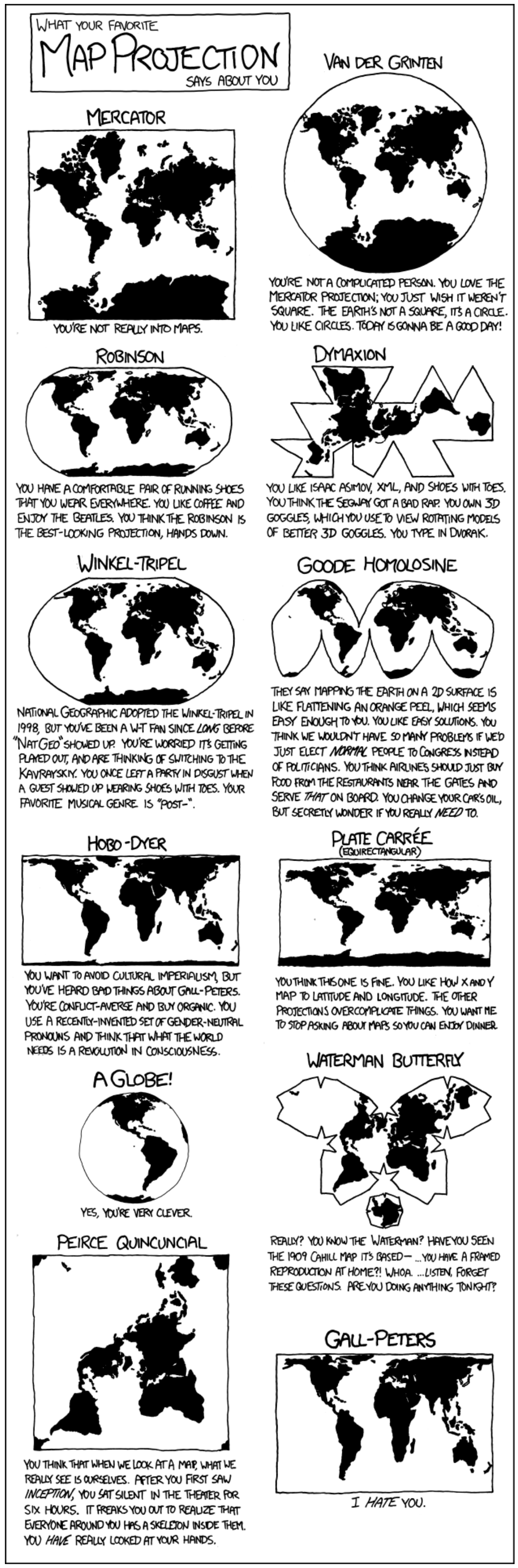
Another piece of good news is that it is possible to preserve the area of all countries and continents at once. However, angles or distances cannot be preserved in this case.

6.3 Preserving Distances to One (or Two) Given Point(s)

It is not possible to preserve the distance between all pairs of points on Earth, but it is possible, once points have been fixed, to produce a map preserving distances to the two given points.

6.4 Every Map of the Earth or Its Parts Reflects a Choice

Now that we have seen that there is no faithful map and that it is generally not possible to produce an optimal map, even just in terms of distances. Each map therefore makes compromises that are not always trivial.

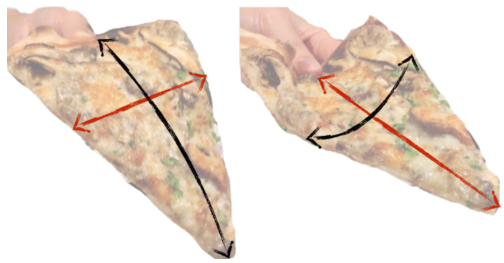


WHAT YOUR FAVORITE
MAP PROJECTION
SAYS ABOUT YOU
MERCATOR
YOU'RE NOT REALLY INTO MAPS.
VAN DER GRINTEN
YOU'RE NOT A COMPLICATED PERSON. YOU LOVE THE MERCATOR PROJECTION; YOU JUST WISH IT WEREN'T SQUARE. THE EARTH'S NOT A SQUARE, IT'S A CIRCLE. YOU LIKE CIRCLES. TODAY IS GONNA BE A GOOD DAY!
ROBINSON
YOU HAVE A COMFORTABLE PAIR OF RUNNING SHOES THAT YOU WEAR EVERYWHERE. YOU LIKE COFFEE AND ENJOY THE BEATLES. YOU THINK THE ROBINSON IS THE BEST-LOOKING PROJECTION, HANDS DOWN.
DYMAXION
YOU LIKE ISAAC ASIMOV, XML, AND SHOES WITH TOES. YOU THINK THE SEGWAY GOT A BAD RAP. YOU OWN 3D GOGGLES, WHICH YOU USE TO VIEW ROTATING MODELS OF BETTER 3D GOGGLES. YOU TYPE IN DVORAK.
WINKEL-TRIPEL
NATIONAL GEOGRAPHIC ADOPTED THE WINKEL-TRIPEL IN 1998, BUT YOU'VE BEEN A W-T FAN SINCE LONG BEFORE "NATGEO" SHOWED UP. YOU'RE WORRIED IT'S GETTING PLAYED OUT, AND ARE THINKING OF SWITCHING TO THE KAVRAYSKIY. YOU ONCE LEFT A PARTY IN DISGUST WHEN A GUEST SHOWED UP WEARING SHOES WITH TOES. YOUR FAVORITE MUSICAL GENRE IS "POST-".
GOODE HOMOLOSINE
THEY SAY MAPPING THE EARTH ON A 2D SURFACE IS LIKE FLATTENING AN ORANGE PEEL, WHICH SEEMS EASY ENOUGH TO YOU. YOU LIKE EASY SOLUTIONS. YOU THINK WE WOULDN'T HAVE SO MANY PROBLEMS IF WE'D JUST ELECT NORMAL PEOPLE TO CONGRESS INSTEAD OF POLITICIANS. YOU THINK AIRLINES SHOULD JUST BUY FOOD FROM THE RESTAURANTS NEAR THE GATES AND SERVE THAT ON BOARD. YOU CHANGE YOUR CAR'S OIL, BUT SECRETLY WONDER IF YOU REALLY NEED TO.
HOBO-DYER
YOU WANT TO AVOID CULTURAL IMPERIALISM, BUT YOU'VE HEARD BAD THINGS ABOUT GALL-PETERS. YOU'RE CONFLICT-AVERSE AND BUY ORGANIC. YOU USE A RECENTLY-INVENTED SET OF GENDER-NEUTRAL PRONOUNS AND THINK THAT WHAT THE WORLD NEEDS IS A REVOLUTION IN CONSCIOUSNESS.
PLATE CARRÉE
(EQUIRECTANGULAR)
YOU THINK THIS ONE IS FINE. YOU LIKE HOW X AND Y MAP TO LATITUDE AND LONGITUDE. THE OTHER PROJECTIONS OVERCOMPLICATE THINGS. YOU WANT ME TO STOP ASKING ABOUT MAPS SO YOU CAN ENJOY DINNER.
A GLOBE!
YES, YOU'RE VERY CLEVER.
WATERMAN BUTTERFLY
REALLY? YOU KNOW THE WATERMAN? HAVE YOU SEEN THE 1909 CAHILL MAP IT'S BASED— ...YOU HAVE A FRAMED REPRODUCTION AT HOME?! WHOA. ...LISTEN, FORGET THESE QUESTIONS. ARE YOU DOING ANYTHING TONIGHT?
PEIRCE QUINCUNCIAL
YOU THINK THAT WHEN WE LOOK AT A MAP, WHAT WE REALLY SEE IS OURSELVES. AFTER YOU FIRST SAW INCEPTION, YOU SAT SILENT IN THE THEATER FOR SIX HOURS. IT FREAKS YOU OUT TO REALIZE THAT EVERYONE AROUND YOU HAS A SKELETON INSIDE THEM. YOU HAVE REALLY LOOKED AT YOUR HANDS.
GALL-PETERS
I HATE YOU.

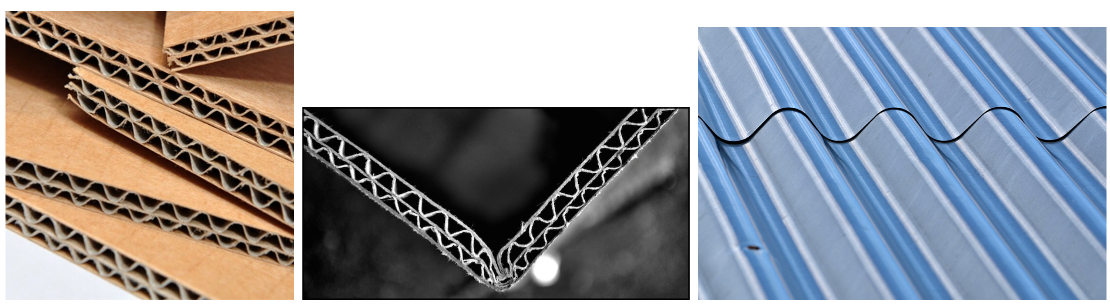
7 Another Application: The Rigidity of Curved Surfaces

Flat surfaces are not rigid because there are deformations that preserve the fact that a geodesic touches the tangent plane at every point. This means that they can be deformed while preserving curvature and thus distances by the Theorema Egregium.

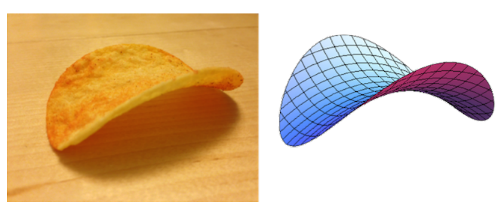
As soon as curvature is imposed in one direction, the other direction is forced to stay straight to touch the surface. The surface becomes rigid.



Some other examples include corrugated materials which are less likely to get bent in one direction. Indeed, as in the case of the pizza, one of the two principal curvatures is forced to be non zero, hence the other is forced to stay straight.



The most common way to induce rigidity is to add curvature to the surfaces under consideration:



8 Further developments

Now curvature has become a central quantity in mathematics and is used in virtually every field.

It is also a central quantity in numerous theories in physics, chemistry or biology. Curvature measures surface tensions, explains the shape of things found in nature, and Einstein's general relativity theory tells us that curvature of space-time is the explanation of gravity.

9 Problem set

9.1 Exercise 1

Consider the surfaces given by the following equations for $(x, y, z) \in \mathbb{R}^3$:

1. $z = x^2 + y^2$,
2. $z = x^2 - y^2$, and
3. $x^2 + y^2 = 1$.

a) Write them as level sets of smooth maps from $\mathbb{R}^3 \to \mathbb{R}$ and compute their tangent space at a point of your choice.

b) What is the sign of the Gauss curvature of each of the surfaces?

c) Give an equation of a corrugated surface which has zero Gauss curvature.

9.2 Exercise 2

Consider a surface of *revolution* given by an equation $f(z) = \sqrt{x^2 + y^2}$ for some $f : \mathbb{R} \to \mathbb{R}$.
a) Write it as a level set Σ_0 of a smooth function from $\mathbb{R}^3 \to \mathbb{R}$, and compute its tangent space.

b) Notice that for every $r > 0$, $r = \sqrt{x^2 + y^2}$ is the radius of a circle in the (x, y) plane. Draw the surfaces obtained (numerically or by hand) for $f_1(r) = 1$, $f_2(r) = r + 1$, $f_3(r) = r^2 + 1$, $f_4(r) = 1 - r^2$ (where $f_i(r) > 0$). From the pictures which ones seem to have positive, negative or zero Gaussian curvature?

c) Determine for which class of function f, the surface obtained has zero curvature.

d) Determine a condition on the second derivative of f under which the surface has everywhere positive Gauss curvature. Same question with negative Gauss curvature.