1. Recall that a ring $R = (R, +, \cdot, -, 0, 1)$ consists of a set $R$ equipped with two binary operations $+$ and $\cdot$, a unary operation $-$, and two constant 0 and 1 satisfying the equational laws:

\[
\begin{align*}
    x + (y + z) &= (x + y) + z, & x \cdot (y \cdot z) &= (x \cdot y) \cdot z, \\
    x + y &= y + x, & x \cdot 1 &= 1 \cdot x = x, \\
    x + 0 &= 0 + x = 0, & x \cdot (y + z) &= x \cdot y + (x \cdot z), \\
    x + (-x) &= (-x) + x = 0, & (y + z) \cdot x &= (y \cdot x) + (z \cdot x).
\end{align*}
\]

Show that in any ring $R = (R, +, \cdot, -, 0, 1)$, 0 is a multiplicative annihilator, i.e., $0 \cdot x = x \cdot 0 = x$ for all $x \in R$.

2. Consider a feed-forward neural network with $n$ input neurons and $m$ output neurons and no hidden layers. The inference step/forward propagation for this network can be written as

\[ I(x) = h(Wx + b) \]

where $W = (w_{i,j})_{i,j}$ is the $m \times n$ weight matrix, $b = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$ is the bias vector, $x \in \mathbb{R}^n$ is our input vector, and $h : \mathbb{R} \to \mathbb{R}$ is our activation function (applied elementwise when applied to a vector). You may assume that $h$ is differentiable.

Let $X$ denote the set of training examples and let $y(x)$ be the ground-truth output associated with $x \in X$. The cost function associated with this network is given by

\[ C = \frac{1}{2n} \sum_{x \in X} |I(x) - y(x)|^2, \]

where $n$ is the size of $X$. Find $\frac{\partial C}{\partial w_{i,j}}$ and $\frac{\partial C}{\partial b_i}$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$.

3. Define

\[ A := \begin{bmatrix} a & b & c \\ 3 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 \end{bmatrix} \quad B := \begin{bmatrix} a & b & c \\ -1 & 1 & 2 \\ 3 & -3 & -1 \\ 1 & 3 & -1 \end{bmatrix} \]

Calculate $A \oplus B$, $A \odot B$, and $A \oplus \odot B$ when $\oplus$ and $\odot$ are given by:

(a) $\oplus = +$, $\odot = \cdot$.

(b) $\oplus = \text{min}$, $\odot = \text{max}$.

(c) $\oplus = \text{max}$, $\odot = +$.

(Note: Be sure to interpret empty entries based on the choice of $\oplus$.)

1