18.095 – Compressed Sensing, Prof. Laurent Demanet – IAP 2024
Problem set

Some useful definitions for $x \in \mathbb{R}^n$:

$$\|x\|_0 = |\{i : x_i \neq 0\}|, \qquad \text{i.e., the number of nonzero components}$$
$$\|x\|_p = \left(\sum_i |x_i|^p \right)^{1/p}, \qquad \text{for } 0 < p < \infty$$
$$\|x\|_\infty = \max_i |x_i|$$

1. Consider the linear system
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 10 \\ 4x_1 + 5x_2 + 6x_3 = 30 \end{cases}$$
Let $x = (x_1, x_2, x_3)$.
   (a) Find the underdetermined least squares solution, i.e., the solution that minimizes $\|x\|_2$.
   (b) Find the solution that minimizes $\|x\|_1$.
   (c) Find all the solutions that minimize $\|x\|_0$.
   (d) Explain how your solution to (c) would change if the second equation had been $4x_1 + 5x_2 + 6x_3 = 20$ instead.

   Justify your answers, but you do not need to show that the solution is unique in parts (a) and (b).

   [Hint: Approach (a) computationally with Python/Julia/Matlab; approach (c) analytically; you can approach (b) either way; and recall that hints need not always be followed.]

2. Show that $\|x\|_{\ell_0} = \lim_{p \to 0} \|x\|_p^p$. [Hint: consider first the simpler case when $n = 1$.]

3. Show that $\min_{x \in \mathbb{R}^n} \|x\|_1$ subject to a single linear constraint $a^T x = b$ always has a one-sparse vector as minimizer. (This is perhaps the simplest illustration of the general phenomenon that solutions of underdetermined ℓ_1 minimization problems tend to be sparse.)

4. A set S is convex when, for all $x_1 \in S$ and $x_2 \in S$, the whole line linking x_1 and x_2 is also in S, i.e., $\lambda x_1 + (1 - \lambda) x_2 \in S$ for all $0 \leq \lambda \leq 1$.
   (a) For $x \in \mathbb{R}^2$, show that $S = \{x : \|x\|_1 \leq 1\}$ is a convex set.
   (b) For $x \in \mathbb{R}^2$, show that $S = \{x : \|x\|_{\frac{1}{2}} \leq 1\}$ is not a convex set.