"Dicey" Polynomials

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January 11, 2021
18.095 IAP Lecture Series

Outline

Review: random variables

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Today, we are only thinking about random variables in the range $\{0, 1, 2, \ldots\}$.

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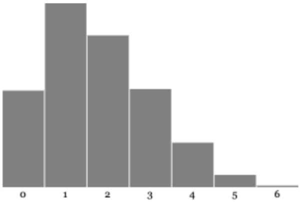

Mean and Variance

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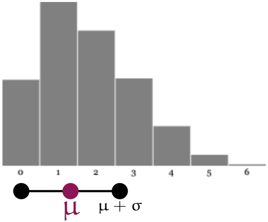

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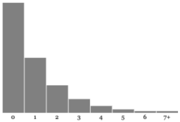
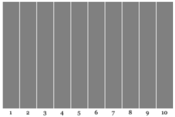
k	$P(X = k)$
0	0.333
1	0.222
2	0.148
3	0.099
4	0.066
5	0.044

Standard Random Variables

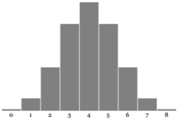
- ***Geometric***
  - “Streak length.”



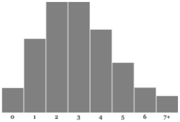
- ***Uniform***
  - Each outcome equally likely.



- ***Binomial***
  - Discrete bell curve.



- ***Poisson***
  - Limiting case of Binomial.



Examples of PGF's

Idea

$$\underbrace{\text{Random Variable } X}_{\text{Probability}} \xleftrightarrow{\text{PGF}} \underbrace{\text{Function } f_X(t)}_{\text{Calculus}}$$

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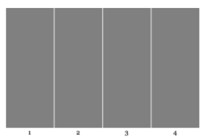
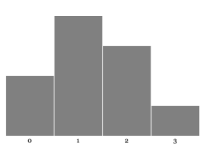


$$f_X(t) = \frac{1}{4} + \frac{1}{4}t + \frac{1}{4}t^2 + \frac{1}{4}t^3$$



$$f_X(t) = \frac{2}{10} + \frac{4}{10}t + \frac{3}{10}t^2 + \frac{1}{10}t^3$$

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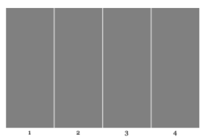

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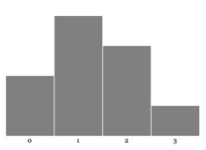

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The function $f_X(t)$ is called the ***probability generating function*** of X.

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We recognize a geometric series:

$$\boxed{f_X(t) = \frac{p}{1 - qt}}$$

Properties of $f_X(t)$

- $f_X(1) =$_____
- $f_X(0) =$__________

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Example: Suppose X is obtained by rolling a six-sided die. Then,

$$f_X(t) = \frac{1}{6}t + \frac{1}{6}t^2 + \cdots + \frac{1}{6}t^6$$

$$f'_X(t) = \frac{1}{6} + \frac{1}{6}2t + \cdots + \frac{1}{6}6t^5$$

$$f'_X(1) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$$

So $E[X] = 7/2 = 3.5$.

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Proof of Theorem:

$$f_X'(t) = \sum_{\text{all } k} kP(X=k)t^{k-1}$$

$$f_X'(1) = \sum_{\text{all } k} kP(X=k) = E[X] \qquad \square$$

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Solution

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\begin{aligned}
f_X(t) &= \frac{p}{1-qt} && \leftarrow \text{from } P(X=k) = q^k p \\
f_X'(t) &= pq(1-qt)^{-2} && \leftarrow \text{chain rule} \\
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Concrete Example: Roll a six-sided die until getting a 4. On average, how many non-4's are rolled? **Answer:** $\frac{5/6}{1/6} = \boxed{5}$.

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Theorem

Let $\mu = f_X'(1) = \mathrm{E}[X]$. Then,

$$\mathrm{Var}(X) = f_X''(1) + \mu(1 - \mu).$$

Proof: Omitted.

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Solution

Compute 2nd derivative:

$f_X'(t) = pq(1-qt)^{-2} \implies f_X''(t) = 2pq^2(1-qt)^{-3}$

Plug in $t = 1$: $f_X''(1) = 2pq^2\underbrace{(1-q)}_{p}^{-3} = \dfrac{2q^2}{p^2}$

Use the Theorem: $\mathrm{Var}(X) = \dfrac{2q^2}{p^2} + \dfrac{q}{p}(1 - \dfrac{q}{p}) = \cdots = \boxed{\dfrac{q^2}{p^2} - \dfrac{q}{p}}$

Let's Get Cooking

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1. $f_X(t) + f_Y(t)$?
2. $f_X(t)f_Y(t)$?
3. $f_X(f_Y(t))$?

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The PGF $\frac{1}{2}f_X + \frac{1}{2}f_Y$ is equal to f_Z where

$$Z = \begin{cases} X \text{ with probability } 1/2 \\ Y \text{ with probability } 1/2 \end{cases}$$

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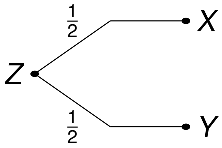

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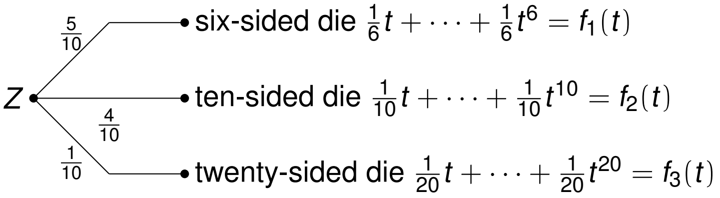

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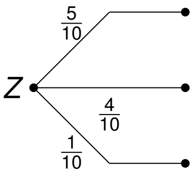

six-sided die $\frac{1}{6}t + \cdots + \frac{1}{6}t^6 = f_1(t)$

ten-sided die $\frac{1}{10}t + \cdots + \frac{1}{10}t^{10} = f_2(t)$

twenty-sided die $\frac{1}{20}t + \cdots + \frac{1}{20}t^{20} = f_3(t)$

Answer: PGF of Z is

$$f_Z = \frac{5}{10}(f_1) + \frac{4}{10}(f_2) + \frac{1}{10}(f_3)$$

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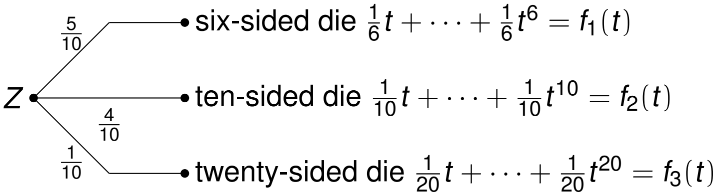

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$$f_Z = \frac{5}{10}(f_1) + \frac{4}{10}(f_2) + \frac{1}{10}(f_3)$$

$$\boxed{f_Z(t) = \frac{\frac{77}{600}t - \frac{1}{200}t^{20} - \frac{1}{25}t^{10} - \frac{1}{12}t^6}{1-t}}$$

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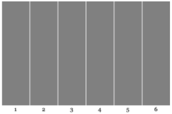
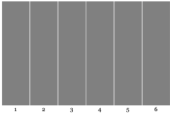
Example:

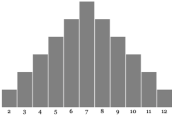
$$f_X = \frac{1}{6}t + \frac{1}{6}t^2 + \frac{1}{6}t^3 + \frac{1}{6}t^4 + \frac{1}{6}t^5 + \frac{1}{6}t^6$$



$$f_Y = \frac{1}{6}t + \frac{1}{6}t^2 + \frac{1}{6}t^3 + \frac{1}{6}t^4 + \frac{1}{6}t^5 + \frac{1}{6}t^6$$



$$f_X f_Y = \frac{1}{36}t^2 + \frac{2}{36}t^3 + \frac{3}{36}t^4 + \frac{4}{36}t^5 + \frac{5}{36}t^6 + \frac{6}{36}t^7 + \frac{5}{36}t^8 + \frac{4}{36}t^9 + \frac{3}{36}t^{10} + \frac{2}{36}t^{11} + \frac{1}{36}t^{12}$$



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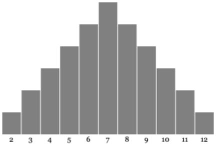

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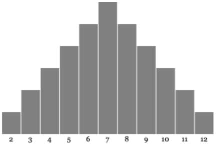

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

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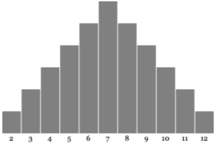

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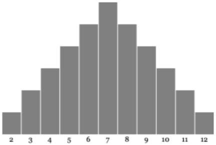

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Proof: The coefficient of x^n in $\left(\sum_{\text{all } k} P(X=k)x^k\right)\left(\sum_{\text{all } k} P(Y=k)x^k\right)$ is $\sum_{a+b=n} P(X=a)P(Y=b)$. □

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The PGF of each shot is $(q + pt)$.

Therefore, $f_X(t) = \underbrace{(q + pt)(q + pt)\ldots(q + pt)}_{n \text{ times}} = (q + pt)^n$

Archery Contest $\sim$ Binomial

Archery Contest ∼ Binomial

The Binomial(n, p) distribution arises from making n independent attempts at a probability p event. We have shown the PGF of $X \sim \text{Binomial}(n, p)$ is simply

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Variance

$$\begin{aligned} f_X''(t) &= n(n-1)p^2(q + tp)^{n-2} \\ f_X''(1) &= n(n-1)p^2 \\ \mathrm{Var}(X) &= n(n-1)p^2 + np(1 - np) \\ &= np(p(n-1) + (1 - np)) \\ &= \boxed{npq} \end{aligned}$$

$f_X \circ f_Y$

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Suppose $f_X(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$. Then,

$$f_X(f_Y(t)) = p_0 + p_1(f_Y(t)) + p_2(f_Y(t))^2 + p_3(f_Y(t))^3$$

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Notice: $(f_Y(t))^k$ is the PGF for rolling k Y's and summing.

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Conclusion: $f_X \circ f_Y$ is the PGF for "roll X Y's."

Statement

Theorem ("Random Sum")

Suppose $Y_1, Y_2, \ldots$ are independent identically-distributed random variables $Y_i \sim Y$. Then for any X, the PGF of $Z = \sum_{i=1}^{X} Y_i$ is $f_Z = f_X \circ f_Y$.

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Solution: $f_Z(t) = f_X \circ f_Y$ where $f_X(t) = \frac{1}{6}t + \cdots + \frac{1}{6}t^6$, and $f_Y = \left(\frac{1}{2} + \frac{1}{2}t\right)$.

$$\boxed{f_Z(t) = \frac{1}{6}\left(\frac{1}{2} + \frac{1}{2}t\right) + \cdots + \frac{1}{6}\left(\frac{1}{2} + \frac{1}{2}t\right)^6}$$

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$f'_Z(t) = f'_X(f_Y(t)) \cdot f'_Y(t)$

$f'_Z(1) = f'_X(f_Y(1)) \cdot f'_Y(1) = f'_X(1) \cdot f'_Y(1) = \mathrm{E}[X]\mathrm{E}[Y]$

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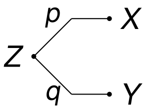

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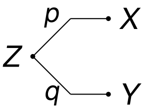

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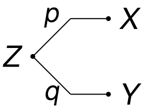

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2. An application

Cheap Trick

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It is also possible to give the exact value of $P(X \text{ is even})$ using just $f_X(-1)$ (see exercises).

Application: Branching Processes

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Setup: Imagine the following population model. We start with one node, and assume that each member of the population decides to have k offspring with probability p_k, for fixed $(p_0, p_1, p_2, \dots)$. This continues indefinitely, or until the population goes extinct.

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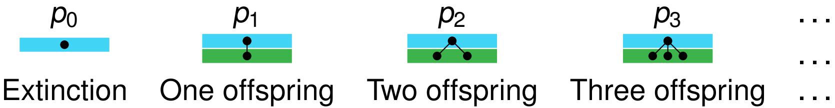
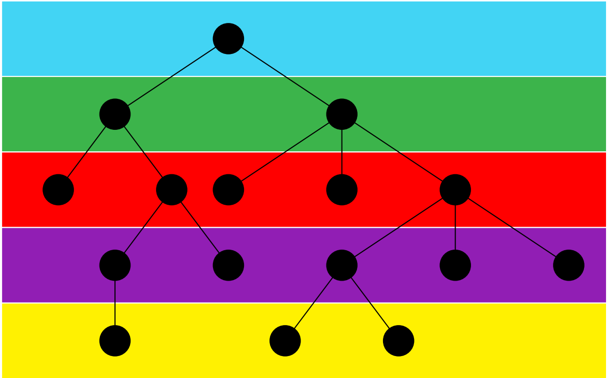


And so on...



Questions:

1. Given the (p_k), what is the probability of eventual extinction?
2. Let G_n be the size of generation k. What is the distribution of G_n?

Applications

1. Population models

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2. **(historical)** Will the names of the aristocracy survive?

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Branching stochastic processes as models of Covid-19 epidemic development

Nikolay M. Yanev[1], Vessela K. Stoimenova[2], Dimitar V. Atanasov[3]

Abstract.
The aim of the paper is to describe two models of Covid-19 infection dynamics. For this purpose a special class of branching processes with two types of individuals is considered. These models are intended to use only the observed daily statistics to estimate the main parameter of the infection and to give a prediction of the mean value of the non-observed population of the infected individuals. Similar problems are considered also in the case when the processes admit an immigration component. This is a serious advantage in comparison with other more complicated models where the officially reported data are not sufficient for estimation of the model parameters. In this way the specific development of the Covid-19 epidemics is considered also for all countries as it is given in the specially created site http://ir-statistics.net/covid-19 where the obtained results are updated daily.

MSC-2020: Primary 92D30
Secondary 60J80; 60J85; 62P10
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Branching Processes Modelling for Coronavirus (COVID'19) Pandemic

Maroussia Slavtchova-Bojkova [1, 2]

[1] Faculty of Mathematics and Informatics, Sofia University, No5, J. Bourchier Blvd., 1164 Sofia, Bulgaria
[2] Institute of Mathematics and Informatics. Bulgarian Academy of Sciences
bojkova@fmi.uni-sofia.bg

Abstract. The purpose of this paper is to review the recent results in the area of infectious disease modelling using general branching processes. A new simulation method oriented to model the spread of the COVID'19 pandemic caused by SARS-CoV-2 coronavirus is proposed. General branching models turned out to be more appropriate and flexible for describing the spread of an infection in a given population, than discrete time ones. Concretely, Crump-Mode-Jagers branching processes are considered as proper candidates of infectious diseases modelling

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A model of COVID-19 propagation based on a gamma subordinated negative binomial branching process

Jérôme Levesque,[a,b] David W. Maybury,[a,b,*] and R.H.A. David Shaw[a]

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Abstract
Go to:

We build a parsimonious Crump-Mode-Jagers continuous time branching process of COVID-19 propagation based on a negative binomial process subordinated by a gamma subordinator. By focusing on the stochastic nature of the process in small populations, our model provides decision making insight into mitigation strategies as an outbreak begins. Our model accommodates contact tracing and isolation, allowing for comparisons between different types of intervention. We emphasize a physical interpretation of the disease propagation throughout which affords analytical results for comparison to simulations. Our model provides a basis for decision makers to understand the likely trade-offs and consequences between

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	p_0	p_1	p_2	p_3	...

P(extinction)	$\mathbf{1}$	$\mathbf{z}^*$	$(\mathbf{z}^*)^{\mathbf{2}}$	$(\mathbf{z}^*)^{\mathbf{3}}$	...

Therefore, $z^* = p_0 + p_1(z^*) + p_2(z^*)^2 + p_3(z^*)^3 + \dots$

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P(extinction)	$\mathbf{1}$	$\mathbf{z^*}$	$\mathbf{(z^*)^2}$	$\mathbf{(z^*)^3}$	...

Therefore, $z^* = p_0 + p_1(z^*) + p_2(z^*)^2 + p_3(z^*)^3 + \ldots$

RHS is the PGF of the first generation: $f(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \ldots$. □

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```
 SageMath version 9.1, Release Date: 2020-05-20
 Using Python 3.7.3. Type "help()" for help.

sage: t = var('t')
sage: f(t) = .2 + .5*t + .2*t^2 + .1*t^3
sage: solve(f(t)==t,t)
[t == -1/2*sqrt(17) - 3/2, t == 1/2*sqrt(17) - 3/2, t == 1]
sage: (1/2*sqrt(17)-3/2).n()
0.561552812808830
```

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Theorem

For $n \geqslant 1$, the PGF of G_n is

$$\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$$

Proof: Induction.

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Proposition

The sequence

$$f(0), f(f(0)), f(f(f(0))), \dots$$

is monotonically increasing and converges to $P(\text{extinction})$.

Proof: Omitted.

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Corollary 1

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$$E[G_n] = \mu^n.$$

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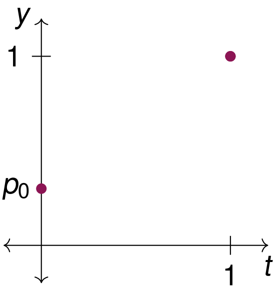
As long as $p_0 > 0$ then we have $P(\text{extinction}) > 0$.

Picture of $y = f(t)$

$$f(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \dots$$



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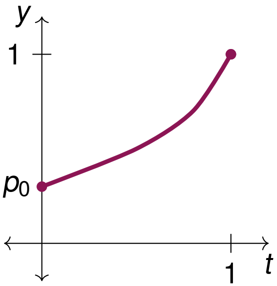

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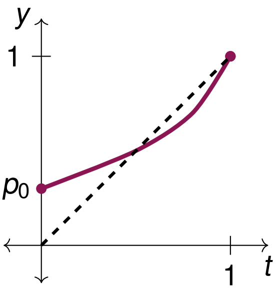

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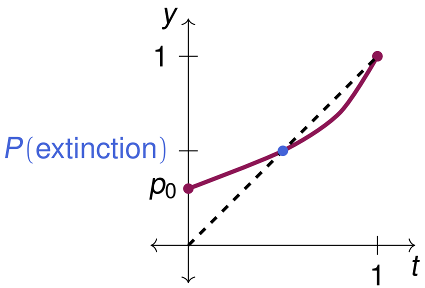

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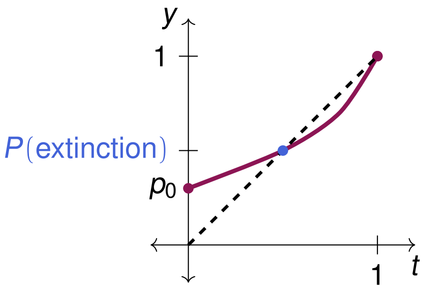

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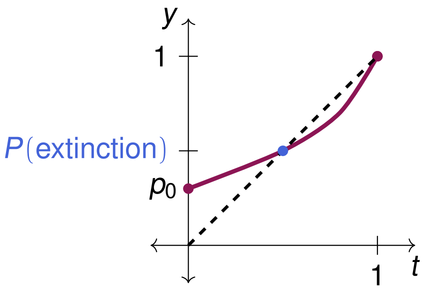

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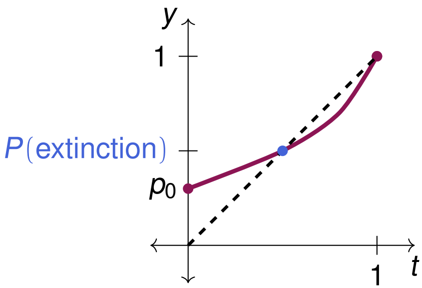

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Thank you!