NAME:

Probability Generating Function Problem Set

18.095 IAP WINTER 2021 LEVEAR

Evens and Odds Suppose X is a discrete random variable with probability generating function $p_X(t)$.

(1) Suppose $p_X(-1) > 0$. What does that tell you about $P(X \text{ is even})$?

(2) Suppose $p_X(-1) = g$. Find a formula for $P(X \text{ is even})$ in terms of g. *Hint*: solve a linear system where the unknowns are $P(X \text{ is even})$ and $P(X \text{ is odd})$.

(3) Consider the game: roll a six-sided die until getting a 4. Let X be the number of non-4's rolled. Find $P(X \text{ is even})$.

Poisson Given $\lambda > 0$, a Poisson(λ) random variable X has distribution $P(X = k) = c\frac{\lambda^k}{k!}$, where c is a constant such that $\sum_{k=0}^{\infty} P(X = k) = 1$.

(1) What is the value of c?

(2) Find a compact expression for the PGF of X.

(3) Find $E[X]$ and $\text{Var}(X)$.

(4) A surprising fact about Poissons is that the sum of two Poissons is a Poisson. Show that if $X \sim \text{Poisson}(\lambda = \rho_1)$ and $Y \sim \text{Poisson}(\lambda = \rho_2)$ then the distribution of $X + Y$ is Poisson with $\lambda = \rho_1 + \rho_2$. Use PGF's!

Poisson Origin One motivation for the Poisson distribution is to approximate Binomials where n is large, since the factorials in binomial coefficients grow very quickly. These next questions guide you through a derivation.

(1) Suppose X is a binomial random variable with parameters n and p. What is the PGF for X?

(2) Now suppose n is large, and write p as λ/n. Re-write the PGF for X using n and λ.

(3) Use an approximation from calculus to conclude that $Y \sim \text{Poisson}(\lambda)$ is a good approximation for X.

Hint: If you don't know what approximation to use, then take a natural log, simplify, and use the linear approximation $\ln(1 + x) \approx x$ for $x \approx 0$.

Poisson Profit You run an online business. You have 100 customers, and each one independently has probability 3% of making a purchase today.

(1) Let X be the total number of sales you make today (=number of purchases made by customers). Then X is a Binomial random variable with what parameters?

(2) A good approximation to X is a Poisson random variable with what parameter? What is the resulting PGF?

(3) Use (2) to approximate the probability of making 5 sales today.

(4) Now suppose each purchase has a 40% chance of making a \$10 profit, a 40% chance of a \$20 profit, and a 20% chance of a \$30 profit.

  (a) What is the PGF for your profit today?

  (b) What is your expected profit today?